Complex Quadratic Polynomials

Definitions (Map(S):, $f^{[n]}$) Let S be a set. We will denote the set of all functions f such that domain(f) = codomain(f) = S by Map(S). We define a function $\mathbf{1}_S$ in Map(S) by

$$\mathbf{1}_S(x) = x \text{ for all } x \in S.$$

If f and g are in Map(S), then the composition $f \circ g$ is in Map(S), so $\circ$ is a binary operation on Map(S). You can easily check that

$$f \circ \mathbf{1}_S = f = \mathbf{1}_S \circ f \text{ for all } f \in \text{Map}(S),$$

so $\mathbf{1}_S$ is an identity element for $\circ$. We will call $\mathbf{1}_S$ the *identity function*, or *identity map* for S.

Let f, g, and h be elements of Map(S). Then for all $x \in S$, we have

$$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))) = f((g \circ h)(x)) = (f \circ (g \circ h))(x).$$

It follows that $f \circ (g \circ h)) = (f \circ g) \circ h$, i.e. composition is an associative operation on Map(S).

If $f \in \text{Map}(S)$ and $n \in \mathbf{N}$, we define $f^{[n]} \in \text{Map}(S)$ by

$$\begin{aligned} f^{[0]} &= \mathbf{1}_S. \\ f^{[n+1]} &= f \circ f^{[n]} \text{ for all } n \in \mathbf{N}. \end{aligned}$$

Thus

$$\begin{aligned} f^{[1]} &= f \circ f^{[0]} = f \circ \mathbf{1}_S = f. \\ f^{[2]} &= f \circ f^{[1]} = f \circ f. \\ f^{[3]} &= f \circ f^{[2]} = f \circ f \circ f. \\ &\cdots \end{aligned}$$

You can show by induction that

$$f^{[n]} \circ f^{[m]} = f^{[n+m]} \text{ for all } m, n \in \mathbf{N}.$$

Definition (orbit): Let S be a set, let $f \in \text{Map}(S)$, and let $a \in S$. The *orbit of a under f*, is the sequence

$$O(f, a) = \{f^{[n]}(a)\} = \{a, f(a), f(f(a)), f(f(f(a))), \cdots\}.$$

We say that the orbit of a under f is *bounded*, if $O(f, a)$ is a bounded sequence.

Examples: Let $c \in \mathbf{C}$. Define functions t_c, r_c, m_c, and s in Map($\mathbf{C}$) by

$$t_c(z) = c + z \text{ for all } z \in \mathbf{C}. \quad (1)$$

$$r_c(z) = c - z \text{ for all } z \in \mathbf{C}. \quad (2)$$

$$m_c(z) = cz \text{ for all } z \in \mathbf{C}. \quad (3)$$

$$s(z) = z^2 \text{ for all } z \in \mathbf{C}. \quad (4)$$

Then

$$
\begin{aligned}
O(t_c, a) &= \{a, c+a, 2c+a, 3c+a, \cdots\} = \{nc+a\}. \\
O(r_c, a) &= \{a, c-a, a, c-a, a, c-a, \cdots\}. \\
O(m_c, a) &= \{a, ca, c^2a, c^3a, \cdots\} = \{c^n a\}. \\
O(s, a) &= \{a, a^2, a^4, a^8, \cdots\} = \{a^{(2^n)}\}.
\end{aligned}
$$

Thus

$$(O(t_c, a) \text{ is bounded}) \iff (c = 0).$$

$$O(r_c, a) \text{ is bounded for all } a \in \mathbf{C} \text{ and all } c \in \mathbf{C}.$$

$$(O(m_c, a) \text{ is bounded}) \iff (a = 0 \text{ or } |c| \leq 1).$$

$$(O(s, a) \text{ is bounded}) \iff (|a| \leq 1).$$

Definition (fixed point): Let S be a set, let $f \in \text{Map}(S)$, and let $a \in S$. We say that a is a *fixed point for* f if $f(a) = a$. The set of all fixed points for f is denoted by $\text{Fix}(f)$.

Examples: For the functions defined in (1) - (4) we have

$$
\begin{aligned}
\text{Fix}(t_c) &= \begin{cases} \mathbf{C} & \text{if } c = 0 \\ \emptyset & \text{if } c \neq 0. \end{cases} \\
\text{Fix}(r_c) &= \{\frac{c}{2}\}. \\
\text{Fix}(m_c) &= \begin{cases} \{0\} & \text{if } c \neq 1 \\ \mathbf{C} & \text{if } c = 1. \end{cases} \\
\text{Fix}(s) &= \{0, 1\}.
\end{aligned}
$$

Note that a is a fixed point for f if and only if the orbit for a under f is the constant sequence $\tilde{a} = \{a, a, a, \cdots\}$.

Example Let $\lambda \in \text{Map}(\mathbf{R})$ be defined by

$$\lambda(x) = x^2 - 1 \text{ for all } x \in \mathbf{R}.$$

Then

$$O(\lambda, \sqrt{2}) = \{\sqrt{2}, 1, 0, -1, 0, -1, \cdots\}.$$

$$O(\lambda, \tfrac{3}{2}) = \{\tfrac{3}{2}, \tfrac{5}{4}, \tfrac{9}{16}, -\tfrac{175}{256}, \cdots\}$$

From this we see that $O(\lambda, \sqrt{2})$ is bounded, but it is not clear whether or not $O(\lambda, \frac{3}{2})$ is bounded.

The following procedure allows you to get an idea of what the orbits of f look like when $f \in \text{Map}(\mathbf{R})$. (In the figure, I've taken $f = \lambda$, and $a = \frac{3}{2}$.)

Recursive procedure for finding orbits for real functions: Let $f \in \text{Map}(\mathbf{R})$, and let $a \in \mathbf{R}$. For $n \in \mathbf{N}$ let

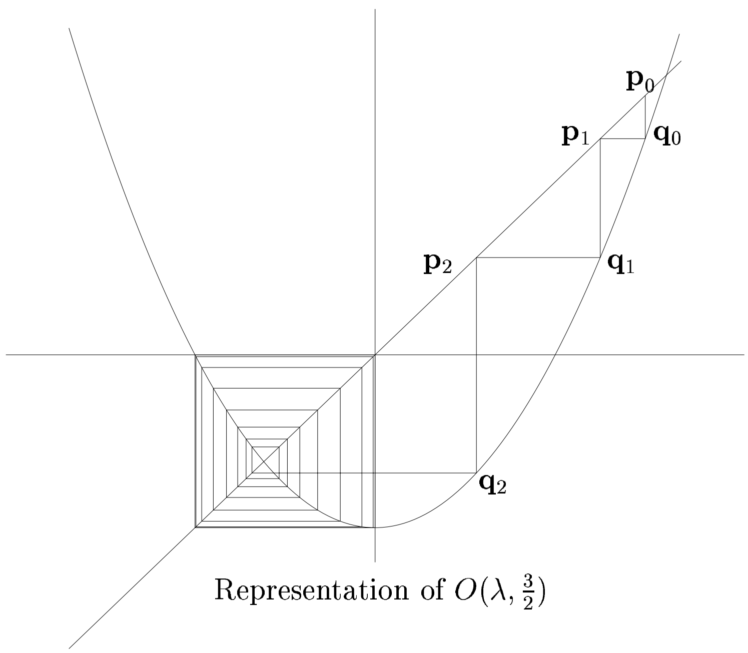
$$
\begin{aligned}
\mathbf{p}_n &= (f^{[n]}(a), f^{[n]}(a)), \\
\mathbf{q}_n &= (f^{[n]}(a), f^{[n+1]}(a)).
\end{aligned}
$$

a) Plot the graphs of f and of $\mathbf{1}_\mathbf{R}$ on the same set of axes. The points where the graphs intersect will satisfy $(x, f(x)) = (x, x)$, i.e. x will be a fixed point for f. Such x are points whose orbits are constant sequences.

b) Mark the point $\mathbf{p}_0 = (a, a)$ on graph$(\mathbf{1}_\mathbf{R})$.

c) For each $n \in \mathbf{N}$ do the following:

  - Draw a vertical line through $\mathbf{p}_n = (f^{[n]}(a), f^{[n]}(a))$ which will intersect graph(f) at $(f^{[n]}(a), f^{[n+1]}(a)) = \mathbf{q}_n$.
  - Draw a horizontal line through $\mathbf{q}_n = (f^{[n]}(a), f^{[n+1]}(a))$, which will intersect graph$(\mathbf{1}_\mathbf{R})$ at $(f^{[n+1]}(a), f^{[n+1]}(a)) = \mathbf{p}_{n+1}$.



Representation of $O(\lambda, \frac{3}{2})$

If we identify the point $(t, t) \in$ graph$(\mathbf{1}_\mathbf{R})$ with the real number t, we get a pretty good idea of what the orbit $O(f, a)$ looks like. From the picture we see that $O(\lambda, \frac{3}{2})$ is bounded, and that for large n the terms $\lambda^{[n]}(\frac{3}{2})$ are alternately very close to -1 and very close to 0.

From the figure it is clear that λ has two fixed points, one of which is between -1 and 0 (call this fixed point β) and the other is a little bigger than $\frac{3}{2}$ (call the larger fixed point α). You can find the exact values of α and β by solving the quadratic equation $\lambda(x) = x$. By applying the procedure just described, convince yourself of the following facts.

i If $a \in [-1, 0]$, then $\lambda^{[n]}(a) \in [-1, 0]$ for all $n \in \mathbf{N}$.

ii If $a \in [0, 1]$, then $\lambda(a) \in [-1, 0]$, so by (i) $O(\lambda, a)$ is bounded.

iii If $a > \alpha$, then $O(\lambda, a)$ is an unbounded increasing sequence.

iv If $a < -\alpha$, then $\lambda(a) > \alpha$, so by (iii) $O(\lambda, a)$ is unbounded. (To see this it may be useful to mark the points $(-\alpha, \alpha)$ and $(-\alpha, -\alpha)$ on the figure above.

v If $a \in (1, \alpha)$, then $O(\lambda, a)$ decreases until some term is in $[0, -1]$, so by (i), $O(\lambda, a)$ is bounded.

vi If $a \in (-\alpha, -1)$, then $\lambda(a) \in (0, \alpha)$, so by (ii) and (v), $O(\lambda, a)$ is bounded.

In summary, $O(\lambda, a)$ is bounded if and only if $a \in [-\alpha, \alpha]$.

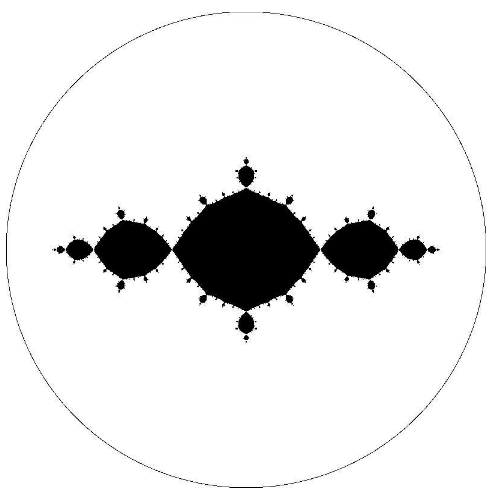
Definition ($J(f), \lambda_c$): Let $f \in \text{Map}(\mathbf{C})$. We define the set $J(f)$ by

$$J(f) = \{a \in \mathbf{C} : O(f, a) \text{ is a bounded sequence}\}.$$

For each $c \in \mathbf{C}$ define $\lambda_c \in \text{Map}(C)$ by

$$\lambda_c(z) = z^2 + c \text{ for all } z \in \mathbf{C}. \tag{5}$$

We will be investigating the sets $J(\lambda_c)$. The discussion above shows that the real numbers in $J(\lambda_{-1})$ form the interval $[-\alpha, \alpha]$, where α is the larger fixed point of λ_{-1}. The complete set $J(\lambda_{-1})a$ is shown below.



$J(\lambda_{-1})$

The circle around this figure is the circle or radius 2 with center at the origin. It is not part of $J(\lambda_{-1})$, and is included just to indicate the scale. Note that the intersection of the figure with the real axis appears to be a line segment centered at the origin, whose half-length appears to be approximately equal to the value of the fixed point α discussed above. Identify where -1 and β and 0 occur in the figure.

We will now discuss how the above figure was obtained.

Lemma: Let $c \in \mathbf{C}$. Then for all $z \in \mathbf{C}$,

$$|z| \geq \left(1 + \sqrt{|c|}\right) \implies |\lambda_c(z)| \geq \left(|z| + \sqrt{|c|}\right). \tag{6}$$

Proof: We have

$$|\lambda_c(z)| = |z^2 + c| \geq |z|^2 - |c| = \Big(|z| - \sqrt{|c|}\Big)\Big(|z| + \sqrt{|c|}\Big).$$

If $|z| \geq 1 + \sqrt{|c|}$, then $\Big(|z| - \sqrt{|c|}\Big) \geq 1$, and hence

$$|\lambda_c(z)| \geq |z| + \sqrt{|c|}. \ \|$$

You can now easily show by induction, that for all $c, z \in \mathbf{C}$, and all $n \in \mathbf{N}$,

$$|z| \geq \Big((1 + \sqrt{|c|}\Big) \implies |\lambda_c^{[n]}(z)| \geq |z| + n\sqrt{|c|}. \tag{7}$$

Corollary: Let $c \in \mathbf{C}$. Then$J(\lambda_c)$ is contained in the disk $D\Big(0, 1 + \sqrt{|c|}\Big)$.
Proof: For all $z \in \mathbf{C}$,

$$z \notin D\Big(0, 1 + \sqrt{|c|}\Big) \implies |z| \geq 1 + \sqrt{|c|} \implies |\lambda_c^{[n]}(z)| \geq |z| + n\sqrt{|c|}.$$

Hence if $c \neq 0$, then

$$z \notin D\Big(0, 1 + \sqrt{|c|}\Big) \implies \{\lambda_c^{[n]}(z)\} \text{ is unbounded} \implies z \notin J(\lambda_c), o \tag{8}$$

and hence

$$z \in J(\lambda_c) \implies \{\lambda[n]_c\} \text{ is bounded} \implies z \in D\Big(0, 1 + \sqrt{|c|}\Big). \tag{9}$$

Thus the corollary follows when $c \neq 0$. If $c = 0$, then

$$J(\lambda_0) = \{z : \{\lambda_0^{[n]}(z)\} \text{ is bounded}\} = \{z : \{z^{2^n}\} \text{ is bounded}\} = D(0,1), \tag{10}$$

so the corollary also holds when $c = 0$. $\|$

To simplify notation and computer programs, we will

ASSUME FOR THE REST OF THIS NOTE THAT $|c| \leq 1$.

Then it follows from our corollary that

$$J(\lambda_c) \subset D(0,2)$$

Notation ($J^n(\lambda_c)$, $I^n(\lambda_c)$): For all $n \in \mathbf{N}$ and all $c \in \mathbf{C}$ with $|c| \leq 1$, we define

$$\begin{aligned} J^n(\lambda_c) &= \{z \in \mathbf{C} : |\lambda_c^{[n]}(z)| \leq 2\}, \\ I^n(\lambda_c) &= \{z \in \mathbf{C} : |\lambda_c^{[n]}(z)| = 2\}. \end{aligned}$$

In particular

$$J^0(\lambda_c) = \{z \in \mathbf{C} : |z| \leq 2\} = \bar{D}(0,2),$$

and

$$I^0(\lambda_c) = \{z \in \mathbf{C} : |z| = 2\} = C(0,2),$$

so $I^0(\lambda_c)$ is the boundary of $J^0(\lambda_c)$. In general, you should think of $I^n(\lambda_c)$ as being the boundary of $J^n(\lambda_c)$. I will often use the curve $I^n(\lambda_c)$ as a representation of the filled-in figure $J^n(\lambda_c)$.

Claim: If $|c| \leq 1$, then

$$J^{[n+1]}(\lambda_c) \subset J^{[n]}(\lambda_c) \text{ for all } n \in \mathbf{N},$$

i.e. the sets $J_n(\lambda_c)$ form a *nested* family of sets.

Proof: By (7),

$$\begin{aligned} z \notin J^n(\lambda_c) &\implies |\lambda_c^{[n]}(z)| > 2 \geq 1 + \sqrt{|c|} \\ &\implies |\lambda^{[n+1]}(z)| = |\lambda(\lambda^{[n]}(z))| \geq |\lambda^{[n]}(z)| + \sqrt{|c|} \geq |\lambda^{[n]}(z)| > 2 \\ &\implies z \notin J^{n+1}(\lambda_c), \end{aligned}$$

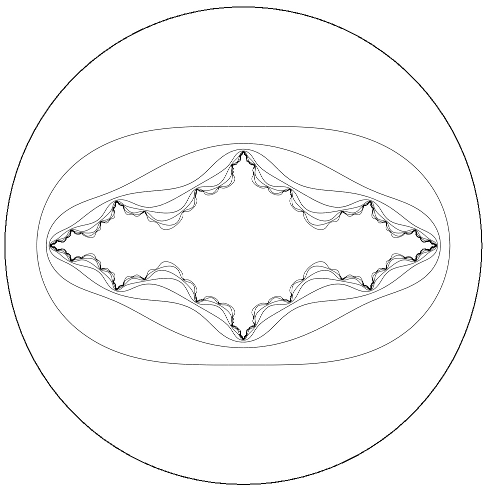
so

$$z \in J^{[n+1]}(\lambda_x) \implies z \in J^n(\lambda_c).$$

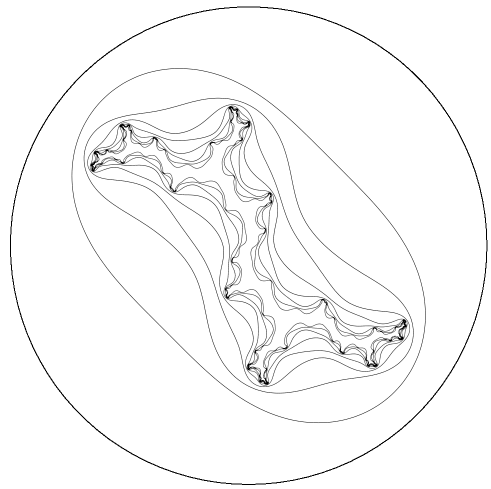
Here are some examples of sets $I^n(\lambda_c)$ (or $J^n(\lambda_c)$) for $0 \leq n \leq 8$ and various values of c. In each case $I^0(\lambda_c)$ is the circle with center 0, and radius equal to 2, and $J^0(\lambda_c)$ is the closed disk with center 0, and radius equal to 2. Also,

$$\cdots J^2(\lambda_c) \subset J^1(\lambda_c) \subset J^0(\lambda_c).$$

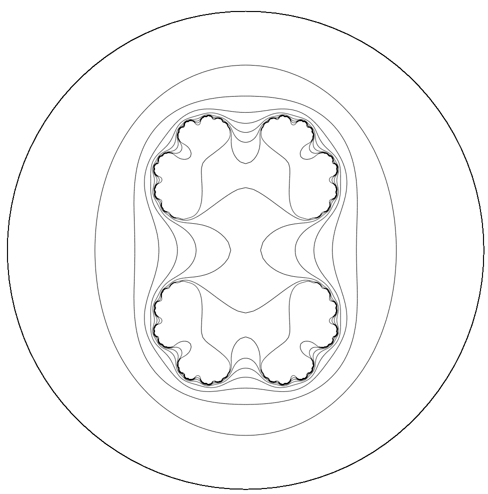
The sets $I^n(\lambda_c)$ (or $J^n(\lambda_c)$)) for $0 \le n \le 8$



$c = -1$



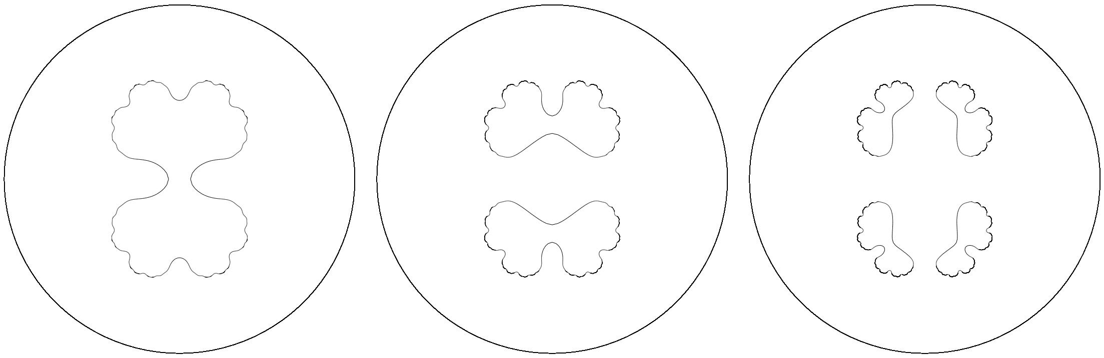
$c = i$



$c = .4$

In the first two cases, each set $I^n(\lambda_c)$ is a connected curve, which you could trace without removing your pen from the paper. However the sets $I^n(\lambda_{.4})$ are not connected when $n \geq 7$, and the individual sets $I^n(\lambda_{.4})$ for $n = 6, 7, 8$ are drawn below.

The sets $I^n(\lambda_{.4})$ for $n = 6, 7, 8$.



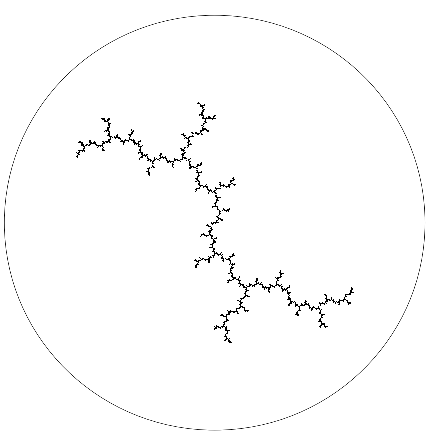
$n = 6$ $n = 7$ $n = 8$

Theorem: Let $c \in \mathbf{C}$ with $|c| \leq 1$. Then for all $z \in \mathbf{C}$

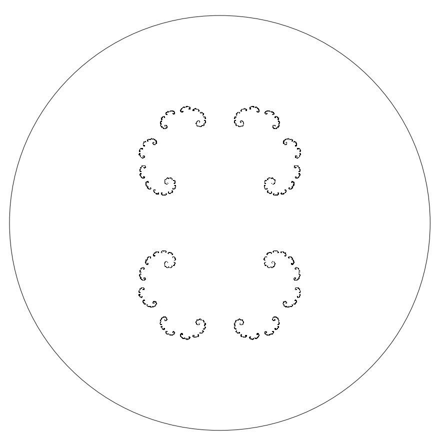
$$(z \in J(\lambda_c)) \iff (z \in J^n(\lambda_c) \text{ for all } n \in \mathbf{N}).$$

Proof: If $z \in J^{[n]}(\lambda_c)$ for all $n \in \mathbf{N}$, then $|\lambda_c^{[n]}(z)| \leq 2$ for all $n \in \mathbf{N}$, so $O(\lambda_c, z)$ is bounded, and hence $z \in J(\lambda_c)$. Conversely, if $z \notin J^{[p]}(\lambda_c)$ for some $p \in \mathbf{N}$, then $|\lambda_c^{[p]}(z)| > 2$. By (8) (together with the assumption $|c| \leq 1$), it follows that $\{\lambda_c^{[n]}(\lambda_c^{[p]}(z))\} = \{\lambda_c^{[n+p]}(z)\}$ is an unbounded sequence. This sequence is a translate of $\{\lambda^{[n]}(z)\}$, so it follows that $\{\lambda_c^{[n]}(z)\}$ is also unbounded, and $z \notin J(\lambda_c)$. (Here I've used the fact that if a translate of a complex sequence is bounded, then the sequence itself is bounded.)

If n is large, I hope $J^n(\lambda_c)$ is a good approximation to $J(\lambda_c)$. In the examples sketched above, You can probably form a pretty good idea about what $J(\lambda_{-1}))$ and $J(\lambda_i)$ look like, but the shape of $J(\lambda_{.4})$ is less clear. The following pictures give fairly accurate representations for $J(\lambda_i)$ and $J(\lambda_{.4})$. The set $J(\lambda_{-1})$ was shown above.



$J(\lambda_i)$



$J(\lambda_{.4})$

If $|c| \leq 1$, I know that all of the sets $J^{[n]}(\lambda_c)$ are contained in the disc with center 0 and radius 2, which is contained in the Cartesian product $[-2,2]\times[-2,2] \subset \mathbf{R}\times\mathbf{R} = \mathbf{C}$. Let N be a large integer (for the figures in this note, I always take $N = 200$). For $-2N \leq i, j \leq 2N-1$ let P_{ij} be the square

$$P_{ij} = [\tfrac{i}{N}, \tfrac{i+1}{N}] \times [\tfrac{j}{N}, \tfrac{j+1}{N}].$$

I will call the squares P_{ij} *pixels* (for *picture elements*). Note that

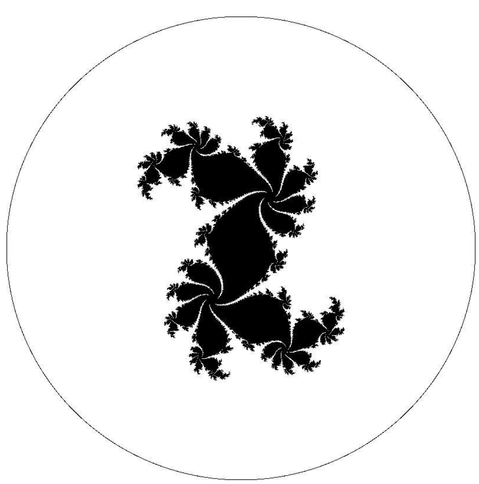
$$[-2,2] \times [-2,2] = \bigcup\{P_{ij} : -2N \leq i, j \leq 2N-1\}.$$

The midpoint of the pixel P_{ij} is the point

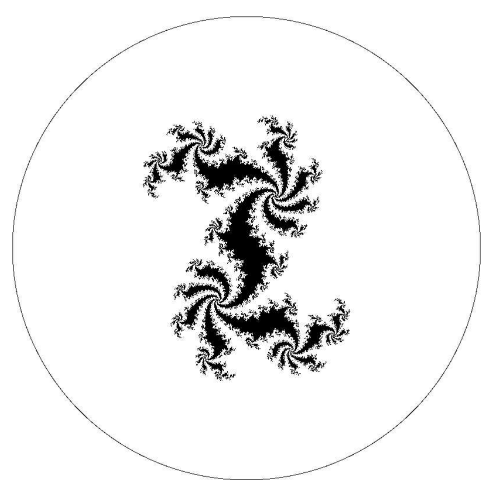
$$c_{ij} = \left(\frac{i+\frac{1}{2}}{N}, \frac{j+\frac{1}{2}}{N}\right).$$

To draw my picture of $J^{[n]}(\lambda_c)$, I do the following:

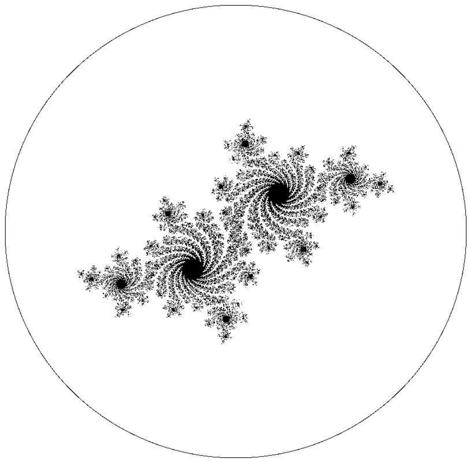
For each pair (i, j) with $-2N \leq i, j \leq 2N-1$, I calculate $\lambda_c^{[n]}(c_{ij})$. If $|\lambda_c^{[n]}(c_{ij})| \geq 2$, then $c_{ij} \notin J^n(\lambda_c)$, and in this case I color the pixel P_{ij} white. If $\lambda_c^{[n]}(c_{ij}) < 2$ then $c_{ij} \in J^n(\lambda_c)$, and in this case I color P_{ij} black. All of my calculations are done on a computer that rounds off numbers to about 17 decimals, so I do not know how accurate my determination of whether $|\lambda_c^{[n]}(c_{ij})| < 2$ is. Also it isn't clear how large to take n to make $J^n(\lambda_c)$ "look like" $J(\lambda_c)$. Also I do not know whether my approximation that only looks at $(4\cdot 200)^2 = 640000$ points really approximates $J(\lambda_c)$ very well. But in any case, the pictures are interesting. Here are some examples.



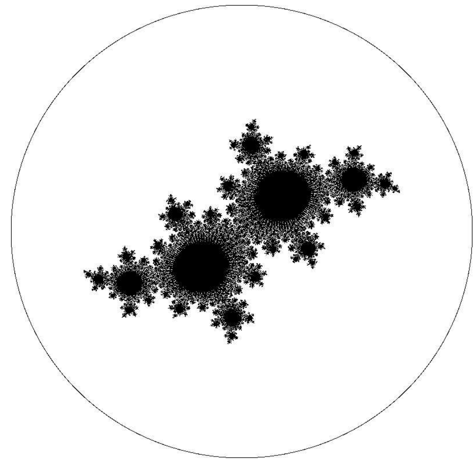
$c = .38 + .22i$, $n = 200$



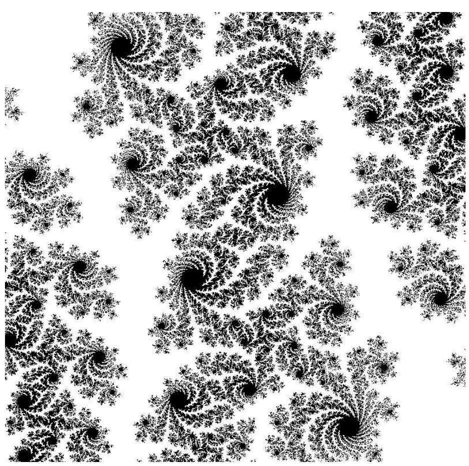
$c = .39 + .23i$, $n = 200$



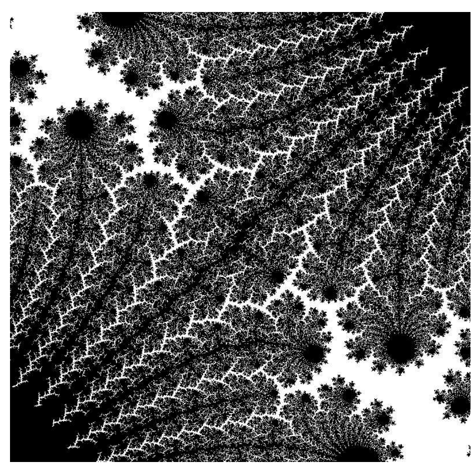
$c = -.4 - .59i$, $n = 200$



$c = -.391 - .587i$, $n = 1000$

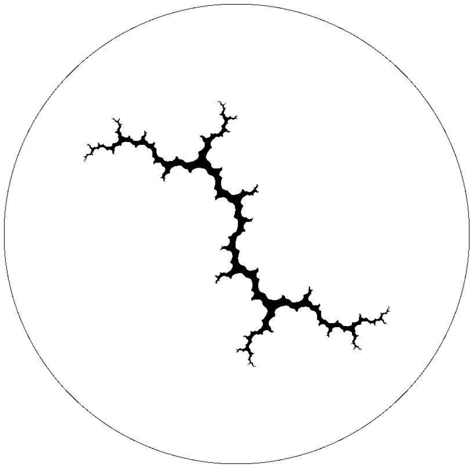


$c = -.4 - .59i$, $n = 200$
$-.2 \leq x, y \leq .2$

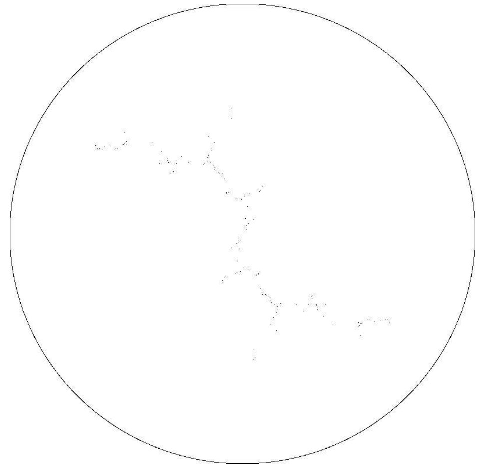


$c = -.391 - .587i$, $n = 1000$
$-.2 \leq x, y \leq .2$

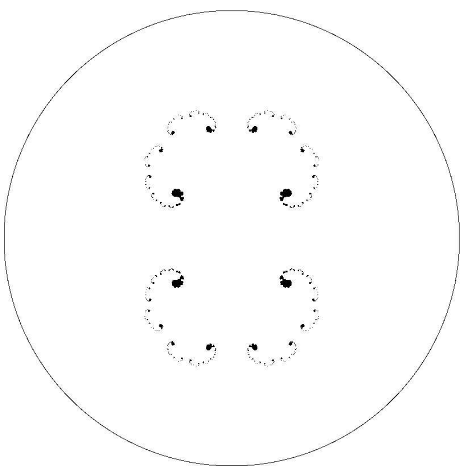
The method just described sometimes gives poor representations for sets $J(\lambda_c)$ that are "thin". The figures below shows the computed values of $J^n(\lambda_c)$ for which the method does not work well.



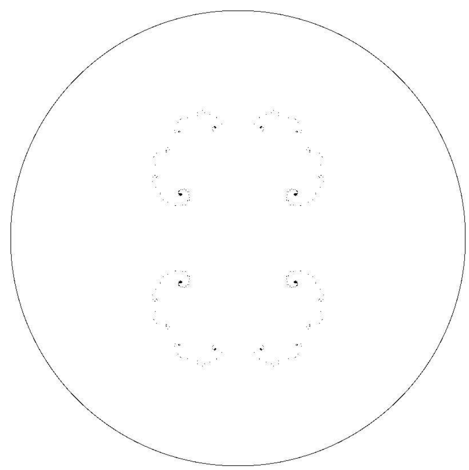
$c = i,\ n = 10$



$c = i,\ n = 25$



$c = .4,\ n = 15$



$c = .4,\ n = 20$

Some properties of $J(\lambda_c)$.

i $J(\lambda_c)$ is symmetric about 0, i.e. for all $z \in \mathbf{C}$,

$$(z \in J(\lambda_c)) \implies (-z \in J(\lambda_c)).$$

Proof: We have $\lambda_c(-z) = \lambda_c(z)$ for all $z \in \mathbf{C}$, so by induction $\lambda_c^{[n]}(-z) = \lambda_c^{[n]}(z)$ for all $n \in \mathbf{Z}_{\geq 1}$. Thus $O(\lambda_c, -z)$ and $O(\lambda_c, z)$ differ only in their first term, and one of these orbits is bounded if and only if the other is. $\|$

ii If c is real, then $J(\lambda_c)$ is symmetric about the real axis, i.e. for all $z \in \mathbf{C}$,

$$(z \in J(\lambda_c)) \implies (z^* \in J(\lambda_c)).$$

Proof: If c is real, then $c = c^*$, so

$$\lambda_c(z^*) = (z^*)^2 + c = (z^*)^2 + c^* = (z^2 + c)^* = (\lambda_c(z))^*.$$

By induction you can show that

$$\lambda_c^{[n]}(z^*) = (\lambda_c^{[n]}(z))^* \text{ for all } n \in \mathbf{N}.$$

Thus $|\lambda_c^{[n]}(z^*)| = |\lambda_c^{[n]}(z)|$ for all $n \in \mathbf{N}$, and $O(\lambda_c, z^*)$ is bounded if and only if $O(\lambda_c, z)$ is bounded $\|$.

iii For all $c \in \mathbf{C}$, $J(\lambda_c)$ is not empty.

Proof: We know that $\text{Fix}(\lambda_c) \subset J(\lambda_c)$. For all $z \in \mathbf{C}$. By the quadratic formula we have

$$z \in \text{Fix}(\lambda_c) \iff z^2 + c = z \iff z = \frac{1 \pm y}{2}$$

where y is a square root of $1 - 4c$. Since all complex numbers have square roots, $\text{Fix}(\lambda_c)$ is never empty, and hence $J(\lambda_c)$ is never empty.

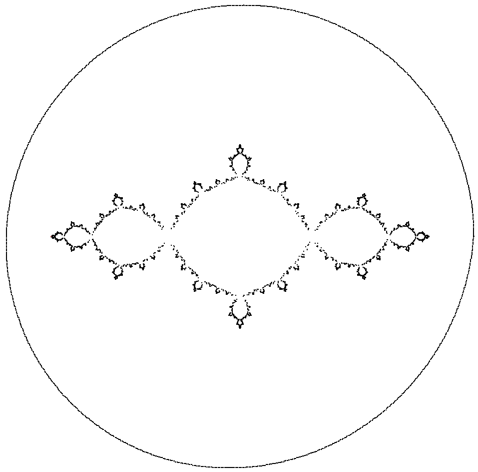
We can find lots of points in $J(\lambda_c)$ as follows. Let α be one of the fixed points of λ_c. We know that α is in $J(\lambda_c)$. Construct a sequence $\{\alpha_n\}$ by the rules

$$\begin{aligned} \alpha_0 &= \alpha \\ \alpha_{n+1} &= \text{one of the solutions of the equation } \lambda_c(z) = \alpha_n. \end{aligned}$$

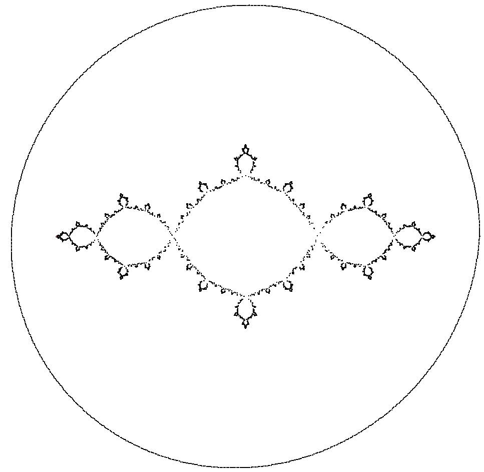
(Thus α_{n+1} is one of the square roots of $\alpha_n - c$.) Then every number α_n is in $J(\lambda_c)$, and in fact $O(\lambda_c, \alpha_n)$ is a sequence that converges to α for every $n \in \mathbf{N}$.

$$\begin{aligned} O(\lambda_c, \alpha_0) &= \{\alpha, \alpha, \alpha, \alpha, \cdots\} \\ O(\lambda_c, \alpha_1) &= \{\alpha_1, \alpha, \alpha, \alpha, \cdots\} \\ O(\lambda_c, \alpha_2) &= \{\alpha_2, \alpha_1, \alpha, \alpha, \cdots\} \\ &\cdots \end{aligned}$$

The figures below show the first 25000 terms for each of the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are the fixed points for λ_{-1}. All of the points in the first figure have orbits that converge to α, and all of the points in the second figure have orbits that converge to β. The two figures appear to be the same. When I plot α_n I am actually drawing the pixel that contains α_n, and similarly for β_n. Most pixels that contain an α_p also contain a β_q. In the process of plotting 25000 points, many pixels have been colored many times. I used a random number generator to decide whether α_{n+1} should be the square root of $\alpha_n - c$ having positive or negative real part.



Points in $\{\alpha_n\}$



Points in $\{\beta_n\}$.

You might notice that in the first figure there are no points near $\pm\beta$, while in the second figure there are points near $\pm\beta$.

If z is a complex number such that the sequence $O(\lambda_c, z)$ converges, then $\lim O(\lambda_c, z)$ must be a fixed point for λ_c. For suppose $\{\lambda_c^{[n]}(z)\} \to L$. Then by the translation theorem,

$$\{\lambda_c(\lambda_c^{[n]}(z))\} = \{\lambda_c^{[n+1]}(z)\} \to L.$$

Since λ_c is a continuous function, it folows that

$$L = \lim\{\lambda_c^{[n+1]}(z)\} = \lim\{\lambda_c(\lambda_c^{[n]}(z))\} = \lambda_c(L),$$

i.e. $L \in \mathrm{Fix}(\lambda_c)$.

Notes: The sets $J(\lambda_c)$ are closely related to sets called *Julia Sets*, named after Gaston Julia (1893–1978), who studied their properties around 1918 (with no computer graphics).

Julia sets can be defined for any function $f \in \mathrm{Map}(\mathbf{C})$. Some beautiful julia sets for rational functions that are not quadratic polynomials can be found at

`http://www.ijon.de/mathe/julia/some_julia_sets_1`
`http://www.ijon.de/mathe/julia/some_julia_sets_2`
`http://www.ijon.de/mathe/julia/some_julia_sets_3`
`http://www.ijon.de/mathe/julia/some_julia_sets_4`

Julia sets for functions of the form $f(z) = C\sin(z)$ can be found at

`http://astronomy.swin.edu.au/~pbourke/fractals/sinjulia`

You can draw your own sets $J(\lambda_c)$ at the URL

`http://math.bu.edu/DYSYS/applets/Quadr.html`

You enter the real and imaginary parts of c in the appropriate box, and enter the numer of iterations in the appropriate box, and then press the "compute" button. On my machine this program takes a considerable amount of time to load and to compute.