Introduction

Remembrance of Things Past

You have been working with numbers for most of your life, and you know hundreds of facts about them. The first numbers you encountered were probably the positive integers $1, 2, 3, \cdots$. You learned that multiplication was repeated addition, and that $3 \times 4 = 12$, because 3 groups of apples, each of which contains 4 apples combine to give a group of 12 apples. Later you learned that $\sqrt{12} \cdot \sqrt{12} = 12$. This did not mean that $\sqrt{12}$ groups of apples, each of which contained $\sqrt{12}$ apples combine to give a group of 12 apples. The definition had changed, but whatever it meant, you knew how to get the "right answer". At some point you met decimals. Since $\frac{1}{4} = \frac{25}{100}$, you knew that $\frac{1}{4} = .25$.

However $\frac{1}{3}$ was more of a problem. Although .33333333 was close to $\frac{1}{3}$, the two numbers were not equal. Perhaps you considered *infinite decimals*, so

$$\frac{1}{3} = .3333333 \cdots.$$

Then

$$3 \times \frac{1}{3} = .9999999 \cdots.$$

But $3 \times \frac{1}{3} = 1$, So $.9999999 \cdots$ must be equal to 1. They don't look equal, but .9999999 was probably close enough. At first the fact that $(-1) \times (-1) = +1$ was probably rather puzzling, but you got used to it after a while. You may have encountered the *imaginary number* i such that $i^2 = -1$ and $\sqrt{-a} = \sqrt{a}\, i$ when a is positive. Then you found that

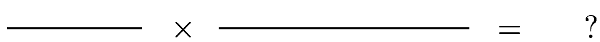
$$\sqrt{-4} \times \sqrt{-9} = \sqrt{(-4)(-9)} = \sqrt{36} = 6,$$

and

$$\sqrt{-4} \times \sqrt{-9} = 2i \cdot 3i = i^2\, 6 = -6.$$

This may have been unnerving. At some time numbers became identified with points on a line. Addition is straightforward: to add two numbers, you just slide the lines so that they share a common end, and combine them into one line.

but what does multiplication mean?



The Goal of the Course

In this course we will reorganize all of the number facts with which you are familiar. We will make a small number of assumptions or *axioms* about numbers, (thirteen assumptions in all, in definitions (2.48), (2.100), and (5.21)). The first twelve assumptions will be familiar number facts. The last assumption may not look familiar, but I hope it will seem as plausible as things you have assumed about numbers in the past. You will not be permitted to assume any facts about numbers other than the thirteen stated assumptions. For example, we will not assume that $3 \cdot 0 = 0$, or that $2 \cdot 2 = 4$, so you will not be allowed to assume this. (These facts will be proved in theorems 2.66 and 2.84.) You will not be allowed to assume that $(-1) \cdot (-1) = 1$, or that $0 < 1$. (These facts will follow from exercise 2.77c and corollary 2.104.) We will not justify the representation of numbers by points on a line, so no proofs can depend on pictures of graphs of functions. On the basis of our assumptions about real numbers, we will construct a more general class of *complex numbers*, in which -1, and in fact every number, has a square root. Many results about the algebra and calculus of real functions will be shown to hold for complex functions.

Occasionally I will draw pictures to motivate proofs, but the proofs themselves will not depend on the pictures. The goal of the course is to "contemplate the very nature of numbers by thought alone, practising it not for the purpose of buying and selling like merchants and hucksters, but $\cdots$ to make easier the change from the world of becoming to real being and truth."

Sometimes in examples or remarks I will use arguments depending on similar triangles or trigonometric identities, but my theorems and definitions will depend only on my assumptions. I will also refer to integers and rational numbers in examples before I give the formal definitions, but no theorems will involve integers until they have been defined. Nothing in this course will be trivial or obvious or clear. If you come across these words, it probably means that I am engaging in a mild deception. Beware.

On page 34 we will prove the well known fact that $2 \cdot 0 = 0$. On page 245, we will prove the less well known fact that $e^{2\pi i} = 1$. The fact that we can derive the last not-so-obvious result from our thirteen assumptions is somewhat remarkable.

Some General Remarks

The **exercises** in these notes are important. The proofs of many theorems will appear as exercises. You should work on as many exercises as time permits. Do not be discouraged if you cannot do some of the exercises the first time you try them. The important thing is that you should be able to do them after they have been discussed in class. The **entertainments** are supposed to be entertaining. If they do not entertain you, you can ignore them (unless your instructor is so entertained that one gets assigned as a homework problem). There are hints for selected problems at the end of the notes. Do not use them until you have spent some time on the problems. Any method you discover on your own is better than any method suggested in a hint.

The prerequisite for this course is a course in one-variable calculus. From the remarks made above you know that you cannot assume any facts from your calculus course, but many theorems are motivated by calculus. You should know that the derivative $f'(a)$ of a function f at a point a is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

and that this represents the slope of the tangent to the graph of f at the point $(a, f(a))$. If you are not familiar with the rules for calculating the derivatives of the sine and cosine and exponential functions from an earlier course, our *definitions* of the sine and cosine and exponential will seem rather meaningless.

In these notes, informal set theory and logic are used. Axioms for set theory and logic can be given, and most mathematicians believe that all of the sorts of informal proofs that we give in the notes can, in principle, be justified by the axioms of set theory and logic. However the sort of informal set theory and logic we use here are typical of the methods used by workers in mathematical analysis at the start of the twenty-first century.

***

These notes are largely based on Joe Buhler's math 112 notes, used at Reed College in spring 1998[13]. The material in Buhler's appendices has been expanded and put into the main text. There are more examples, and some proofs are given in more detail. I've added some pictures, because I think geometrically.

Entertainment 0 Suppose three lines are given, having lengths 1, a, and b.

a) Describe a compass-and-straightedge construction for a line segment of length $a \times b$.

b) Describe a compass-and-straightedge construction for a line segment of length $\sqrt{a}$.