# Chapter 12

# Power Series

## 12.1 Definition and Examples

**12.1 Definition (Power Series.)** Let $\{a_n\}$ be a sequence of complex numbers. A series of the form $\sum\{a_n z^n\}$ is called a *power series.*

We think of a power series as a sequence of polynomials

$$\{a_0, a_0 + a_1 z, a_0 + a_1 z + a_2 z^2, a_0 + a_1 z + a_2 z^2 + a_3 z^3, \cdots\}.$$

In general, this sequence will converge for certain complex numbers, and diverge for other numbers. A power series $\sum\{a_n z^n\}$ determines a function whose domain is the set of all $z \in \mathbf{C}$ such that $\sum\{a_n z^n\}$ converges.

**12.2 Examples.** The geometric series $\sum\{z^n\}$ is a power series that converges to $\dfrac{1}{1-z}$ for $|z| < 1$ and diverges for $|z| \geq 1$.

The series $C = \sum\left\{\dfrac{(-1)^n z^{2n}}{(2n)!}\right\}$ and $S = \sum\left\{\dfrac{(-1)^n z^{2n+1}}{(2n+1)!}\right\}$ are power series that converge for all $z \in \mathbf{C}$. $C$ corresponds to the sequence

$$\{a_n\} = \left\{1, 0, -\frac{1}{2}, 0, \frac{1}{24}, \cdots\right\}$$

and $S$ corresponds to

$$\{a_n\} = \left\{0, 1, 0, -\frac{1}{6}, 0, \frac{1}{120}, \cdots\right\}.$$

The limits are $\cos z$ and $\sin z$, respectively (by definition 11.43.)

Every power series $\sum\{a_n z^n\}$ converges at $z = 0$. (The limit is $a_0$.)

The series $\sum\{n! z^n\}$ converges only when $z = 0$ (see exercise 12.5).

**12.3 Notation ($a^{b^c}$)** The expression $a^{b^c}$ is ambiguous. Since

$$2^{(2^3)} = 2^8 = 256,$$

and

$$(2^2)^3 = 4^3 = 64,$$

we see that in general $a^{(b^c)} \neq (a^b)^c$. We make the convention that

$$a^{b^c} \text{ means } a^{(b^c)}.$$

The expression $(a^b)^c$ is usually simplified and written without parentheses by use of theorem 3.64:

$$(a^b)^c = a^{(bc)} = a^{bc}.$$

**12.4 Example.** I would like to consider the series $\sum\left\{\frac{z^{n^2}}{n^2}\right\}_{n\geq 1}$ to be a power series. This series corresponds to $\sum\{c_n z^n\}$ where

$$\begin{aligned} \{c_n\} &= \{0, 1, 0, 0, \frac{1}{4}, 0, 0, 0, 0, \frac{1}{9}, \cdots\} \\ \sum\{c_n z^n\} &= \{0, z, z, z, z + \frac{z^4}{4}, z + \frac{z^4}{4}, \cdots\}, \end{aligned}$$

which is not identical with

$$\sum\left\{\frac{z^{n^2}}{n^2}\right\}_{n\geq 1} = \left\{z, z + \frac{z^4}{4}, z + \frac{z^4}{4} + \frac{z^9}{9}, \cdots\right\},$$

but you should be able to see that one series converges if and only if the other does, and that they have the same limits. In the future I will sometimes blur the distinctions between two series like this.

For $z \neq 0$, let $a_n = \frac{z^{n^2}}{n^2}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{z^{n^2+2n+1}}{(n+1)^2}\right| \left|\frac{n^2}{z^{n^2}}\right| = |z|^{2n+1}\left(\frac{n}{n+1}\right)^2.$$

If $|z| < 1$, then $|z|^{2n+1}\left(\dfrac{n}{n+1}\right)^2 \leq |z|^{2n+1}$ and $\lim\left\{\dfrac{|a_{n+1}|}{|a_n|}\right\}_{n\geq 1} = 0 < 1$, so by the ratio test, $\sum\left\{\dfrac{z^{n^2}}{n^2}\right\}_{n\geq 1}$ converges absolutely for $|z| < 1$.

If $|z| > 1$ and $n \geq 1$, then

$$\left\{\left|\frac{a_{n+1}}{a_n}\right|\right\} = |z|^{2n+1}\left(1 - \frac{1}{(n+1)}\right)^2 \geq |z|^{2n+1}\cdot\frac{1}{4},$$

so $\dfrac{|a_{n+1}|}{|a_n|} > 1$ for large $n$, and the series diverges. If $|z| = 1$, then $|a_n| = \dfrac{1}{n^2}$, so $\sum\{|a_n|\}$ converges by the comparison test, and $\sum\left\{\dfrac{z^{n^2}}{n^2}\right\}_{n\geq 1}$ converges absolutely. This shows that the function

$$f(z) = \sum_{n=1}^{\infty}\frac{z^{n^2}}{n^2}$$

is defined for all $z \in \bar{D}(0,1)$, and determines a function from $\bar{D}(0,1)$ into $\mathbf{C}$. The figure on page 229 shows the images under $f$ of circles of radius $\dfrac{j}{10}$ for $1 \leq j \leq 10$ and of rays that divide the disc into twelve equal parts. The images of the interior circles are nice differentiable curves. The image of the boundary circle seems to have interesting properties that I do not know how to demonstrate.

**12.5 Exercise.**

a) Show that $\sum\{n!z^n\}$ converges only for $z = 0$.

b) Show that $\sum\left\{\dfrac{z^{2^n}}{2^n}\right\}$ converges if and only if $|z| \leq 1$. |||

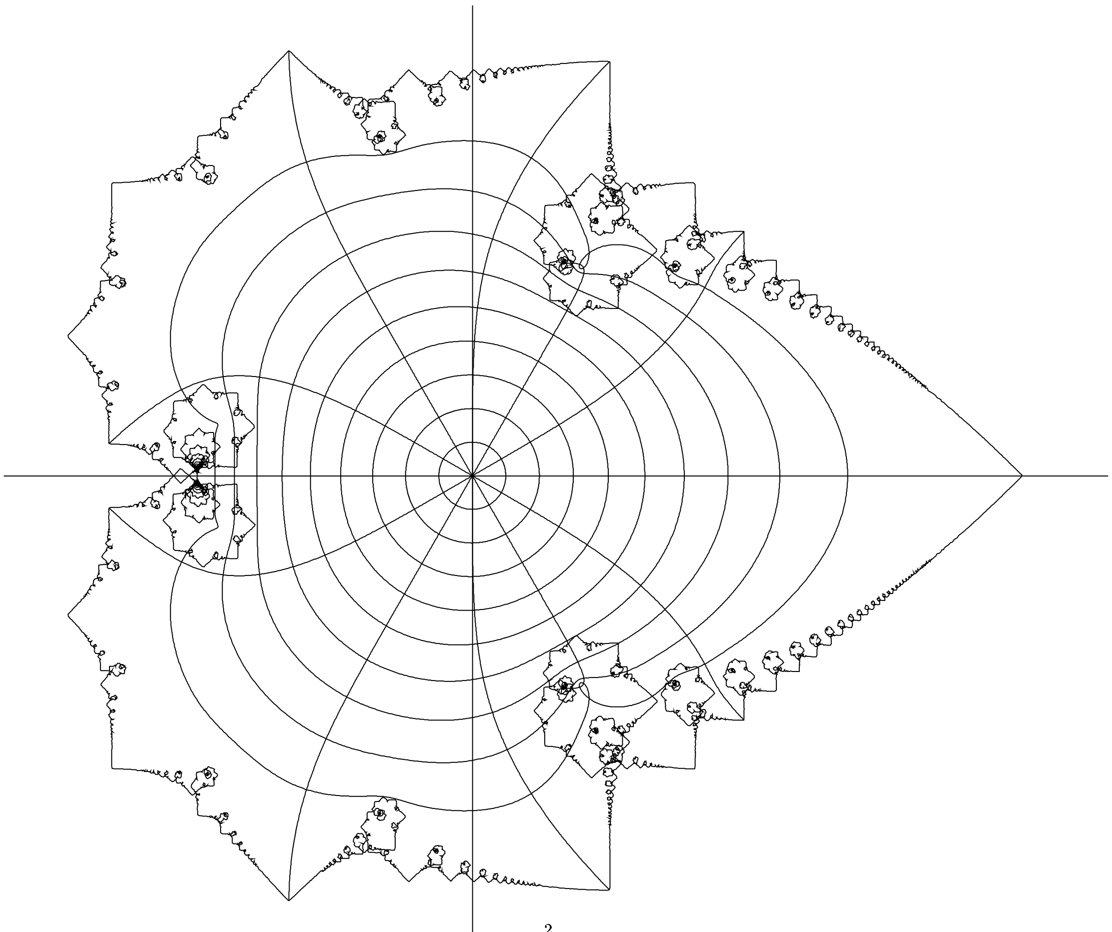


Image of $\sum\left\{\frac{z^{n^2}}{n^2}\right\}_{n\geq 1}$

Let $g(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{2^n}$ for $|z| \leq 1$.

The figure on page 230 shows the images under $g$ of circles of radius $\frac{j}{10}$ for $1 \leq j \leq 10$, and of rays that divide the disc into 12 equal parts.

**12.6 Exercise.** Let $g(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{2^n}$ for $|z| \leq 1$. It appears from figure on page 230 that $g(-1) = 0$, and $g(i)$ is pure imaginary. Show that this is the case.

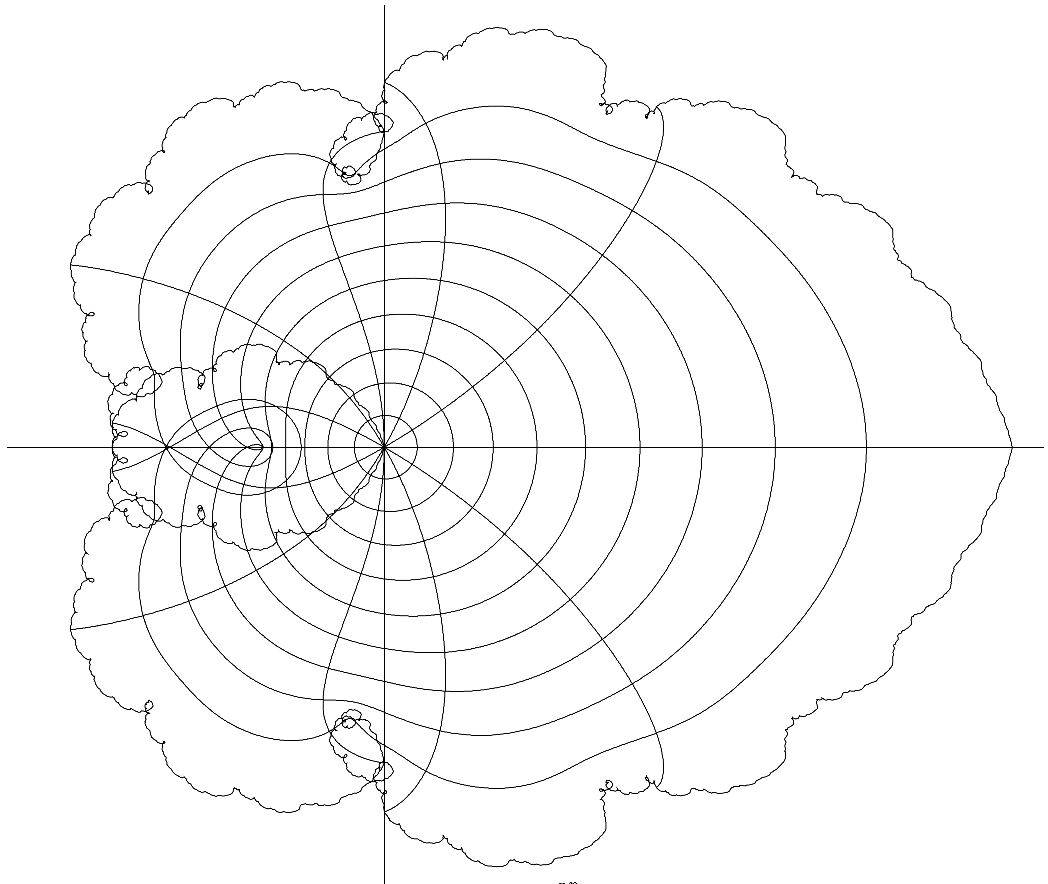


Image of $\sum \left\{ \frac{z^{2^n}}{2^n} \right\}$

**12.7 Entertainment.** It appears from the image of $g(z) = \sum_{n=0}^{\infty} \frac{z^{2^n}}{2^n}$ that if $w = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ (a cube root of $-1$), then $g(w)$ is pure imaginary, and has length a little larger than the length of $g(i)$. Show that this is the case. (From the fact that $w^3 = -1$, notice that

$$\{w^1, w^2, w^4, w^8, w^{16}, w^{32}, w^{64}, \cdots\} = \{w, w^2, -w, w^2, -w, w^2, -w, \cdots\}.)$$

## 12.2 Radius of Convergence

**12.8 Theorem.** *Let $\sum\{a_n z^n\}$ be a power series. Suppose $\sum\{a_n w^n\}$ converges for some $w \in \mathbf{C}\backslash\{0\}$. Then $\sum\{a_n z^n\}$ converges absolutely for all $z \in D(0, |w|)$.*

Proof: Since $\sum\{a_n w^n\}$ converges, $\{a_n w^n\}$ is a null sequence, and hence is bounded. Say $|a_n w^n| \leq M$ for all $n \in \mathbf{N}$. Let $z \in D(0, |w|)$, so $|z| < |w|$, and let $R = \dfrac{|z|}{|w|} < 1$. Then for all $n \in \mathbf{N}$

$$|a_n z^n| = |a_n w^n| \left|\frac{z}{w}\right|^n \leq MR^n.$$

Now $\sum\{MR^n\}$ is a convergent geometric series, so by the comparison test, $\sum\{|a_n z^n|\}$ converges; i.e., $\sum\{a_n z^n\}$ is absolutely convergent. |||

**12.9 Corollary.** *Let $\sum\{a_n z^n\}$ be a power series. Suppose $\sum\{a_n w^n\}$ diverges for some $w \in \mathbf{C}$. Then $\sum\{a_n z^n\}$ diverges for all $z \in \mathbf{C}$ with $|z| > |w|$.*

Proof: Suppose $|z| > |w|$. If $\sum\{a_n z^n\}$ converges, then by the theorem, $\sum\{a_n w^n\}$ would also converge, contrary to our assumption. |||

**12.10 Theorem.** *Let $\sum\{c_n z^n\}$ be a power series. Then one of the following three conditions holds:*

*a) $\sum\{c_n z^n\}$ converges only when $z = 0$.*

*b) $\sum\{c_n z^n\}$ converges for all $z \in \mathbf{C}$.*

*c) There is a number $R \in \mathbf{R}^+$ such that $\sum\{c_n z^n\}$ converges absolutely for $|z| < R$ and diverges for $|z| > R$.*

Proof: Suppose that neither a) nor b) is true. Then there are numbers $w, v \in \mathbf{C}\backslash\{0\}$ such that $\sum\{c_n w^n\}$ converges and $\sum\{c_n v^n\}$ diverges. If $a = \dfrac{|w|}{2}$, and $b = 2|v|$, it follows that $\sum\{c_n a^n\}$ converges and $\sum\{c_n b^n\}$ diverges. By a familiar procedure, build a binary search sequence $\{[a_k, b_k]\}$ such that $[a_0, b_0] = [a, b]$, and for all $k \in \mathbf{N}$, $\sum\{c_n a_k^n\}$ converges and $\sum\{c_k b_k^n\}$ diverges. Let $R$ be the number such that $\{[a_k, b_k]\} \to R$. Then $a_k \leq R \leq b_k$ for all $k \in \mathbf{N}$ and

$\lim\{a_k\} = \lim\{b_k\} = R$.
If $|z| < R$, then for some $k \in \mathbf{N}$ we have $|a_k - R| < R - |z|$, and

$$\begin{aligned} |a_k - R| < R - |z| &\implies a_k > R - (R - |z|) = |z| \\ &\implies \sum\{c_n z^k\} \text{ converges.} \end{aligned}$$

If $|z| > R$, then for some $k \in \mathbf{N}$ we have $|b_k - R| < |z| - R$, and

$$\begin{aligned} |b_k - R| < |z| - R &\implies b_k < R + (|z| - R) = |z| \\ &\implies \sum\{c_n z^n\} \text{ diverges.} \quad \| \end{aligned}$$

**12.11 Definition (Radius of convergence.)** Let $\{\sum c_n z^n\}$ be a power series. If there is a number $R \in \mathbf{R}^+$ such that $\sum\{c_n z^n\}$ converges for $|z| < R$, and diverges for $|z| > R$, we call $R$ the *radius of convergence* of $\sum\{c_n z^n\}$. If $\sum\{c_n z^n\}$ converges only for $z = 0$, we say $\sum\{c_n z^n\}$ has radius of convergence 0. If $\sum\{c_n z^n\}$ converges for all $z \in \mathbf{C}$, we say $\sum\{c_n z^n\}$ has radius of convergence $\infty$.

If a power series has radius of convergence $R \in \mathbf{R}^+$, I call $D(0, R)$ the *disc of convergence* for the series, and I call $C(0, R)$ the *circle of convergence* for the series. If $R = \infty$, I call $\mathbf{C}$ the disc of convergence of the series (even though $\mathbf{C}$ is not a disc).

**12.12 Example.** I will find the radius of convergence for $\sum\left\{\frac{(3n)!z^n}{n!(2n)!}\right\}$. I will apply the ratio test. Since the ratio test applies to positive sequences, I will consider absolute convergence. Let $a_n = \frac{(3n)!z^n}{n!(2n)!}$ for all $n \in \mathbf{N}$. Then for all $z \in \mathbf{C} \setminus \{0\}$,

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{(3(n+1))!|z|^{n+1}}{(n+1)!(2(n+1))!} \cdot \frac{n!(2n)!}{(3n)!|z|^n} = \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(2n+1)(2n+2)}|z| \\ &= \frac{(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})}{(1+\frac{1}{n})(2+\frac{1}{n})(2+\frac{2}{n})}|z|. \end{aligned}$$

Hence

$$\left\{\frac{|a_{n+1}|}{|a_n|}\right\} \to \frac{3 \cdot 3 \cdot 3}{1 \cdot 2 \cdot 2}|z| = \frac{27|z|}{4}.$$

By the ratio test, $\sum\{a_n\}$ is absolutely convergent if $|z| < \frac{4}{27}$, and is divergent if $|z| > \frac{4}{27}$. It follows that the radius of convergence for our series is $\frac{4}{27}$.

**12.13 Exercise.** Find the radius of convergence for the following power series:

a) $\sum\{3^n\sqrt{n}z^n\}_{n\geq 1}$.

b) $\sum\left\{\dfrac{z^n}{n^n}\right\}_{n\geq 1}$.

**12.14 Exercise.** Let $r$ be a positive real number.

a) Find a power series whose radius of convergence is equal to $r$.

b) Find a power series whose radius of convergence is $\infty$.

c) Find a power series whose radius of convergence is 0.

## 12.3 Differentiation of Power Series

If $\sum\{c_n z^n\} = \{c_0, c_0+c_1z, c_0+c_1z+c_2z^2, \cdots\}$ is a power series, then the series obtained by differentiating the terms of $\sum\{c_n z^n\}$ is

$$\sum\{c_n n z^{n-1}\} = \{0, c_1, c_1+2c_2z, c_1+2c_2z+3c_3z^2, \cdots\}.$$

This is not a power series, but its translate

$$\sum\{c_{n+1}(n+1)z^n\} = \{c_1, c_1+2c_2z, c_1+2c_2z+3c_3z^2, \cdots\}$$

is.

**12.15 Definition (Formal derivative.)** If $\sum\{c_n z^n\}$ is a power series, then the *formal derivative of* $\sum\{c_n z^n\}$ is

$$D(\sum\{c_n z^n\}) = \sum\{c_{n+1}(n+1)z^n\}.$$

I will sometimes write $D(\sum\{c_n z^n\}) = \sum\{c_n n z^{n-1}\}$ when I think this will cause no confusion.

**12.16 Examples.**

$$\begin{aligned} D(\sum\{z^n\}) &= \sum\{nz^{n-1}\} = \sum\{(n+1)z^n\} \\ &= \{1, 1+2z, 1+2z+3z^2, \cdots\}. \end{aligned}$$

$$\begin{aligned}
D(S) &= D\left(\sum\left\{\frac{(-1)^n z^{2n+1}}{(2n+1)!}\right\}\right) \\
&= D\left(\left\{0, z, z, z-\frac{z^3}{3!}, z-\frac{z^3}{3!}, z-\frac{z^3}{3!}+\frac{z^5}{5!}, \cdots\right\}\right) \\
&= \left\{1, 1, 1-\frac{z^2}{2!}, 1-\frac{z^2}{2!}, 1-\frac{z^2}{2!}+\frac{z^4}{4!}, \cdots\right\} \\
&= \sum\left\{\frac{(-1)^n z^{2n}}{(2n)!}\right\} = C
\end{aligned}$$

or

$$D\left(\sum\left\{\frac{(-1)^n z^{2n+1}}{(2n+1)!}\right\}\right) = \sum\left\{\frac{(2n+1)(-1)^n z^{2n}}{(2n+1)!}\right\} = \sum\left\{\frac{(-1)^n z^{2n}}{(2n)!}\right\} = C.$$

Our fundamental theorem on power series is:

**12.17 Theorem (Differentiation theorem.)** *Let $\sum\{a_n z^n\}$ be a power series. Then $D(\sum\{a_n z^n\})$ and $(\sum\{a_n z^n\})$ have the same radius of convergence. The function $f$ associated with $\sum\{a_n z^n\}$ is differentiable in the disc of convergence, and the function represented by $D(\sum\{a_n z^n\})$ agrees with $f'$ on the disc of convergence.*

The proof is rather technical, and I will postpone it until section 12.8. I will derive some consequences of it before proving it (to convince you that it is worth proving).

**12.18 Example.** We know that the geometric series $\sum\{z^n\}$ has radius of convergence 1 and $f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$. The differentiation theorem says $D(\sum\{z^n\}) = \sum\{nz^{n-1}\}$ also has radius of convergence 1, and

$$f'(z) = \sum_{n=0}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty}(n+1)z^n \text{ for } |z| < 1;$$

i.e.,

$$\sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2} \text{ for } |z| < 1.$$

We can apply the theorem again and get

$$\sum_{n=0}^{\infty}(n+1)(n)z^{n-1} = \frac{2}{(1-z)^3} \text{ for } |z| < 1,$$

or

$$\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}z^n = \frac{1}{(1-z)^3} \text{ for } |z| < 1.$$

Another differentiation gives us

$$\sum_{n=0}^{\infty}\frac{n(n+1)(n+2)z^{n-1}}{2} = \frac{3}{(1-z)^4} \text{ for } |z| < 1,$$

or

$$\sum_{n=0}^{\infty}\frac{(n+1)(n+2)(n+3)}{3!}z^n = \frac{1}{(1-z)^4} \text{ for } |z| < 1.$$

The pattern is clear, and I omit the induction proof that for all $k \in \mathbf{N}$

$$\begin{aligned} \frac{1}{(1-z)^{k+1}} &= \sum_{n=0}^{\infty}\frac{(n+1)(n+2)(n+3)\cdots(n+k)}{k!}z^n \\ &= \sum_{n=0}^{\infty}\frac{(n+k)!}{n!k!}z^n \text{ for } |z| < 1. \end{aligned}$$

**12.19 Exercise.** By assuming the differentiation theorem, we've shown that the series $\sum\left\{\left(\frac{(n+k)!}{n!k!}\right)z^n\right\}$ has radius of convergence 1 for all $k \in \mathbf{N}$. Verify this directly.

**12.20 Exercise.** Find formulas for $\sum\limits_{n=0}^{\infty} nz^n$ and $\sum\limits_{n=0}^{\infty} n^2z^n$ that are valid for $|z| < 1$. (You may assume the differentiation theorem.)

**12.21 Example.** By the differentiation theorem, if

$$C(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!} \text{ and } S(z) = \sum_{j=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!},$$

then $C$ and $S$ are differentiable on $\mathbf{C}$ and $C'(z) = -S(z)$, and $S'(z) = C(z)$. (We saw in earlier examples that both series have radius of convergence $\infty$, and that the formal derivatives satisfy $DS = C$ and $DC = -S$.) Also, clearly $C(z)$ and $S(z)$ are real when $z$ is real. The discussion in example 10.45 then shows that for real $z$, $C$ and $S$ agree with the cosine and sine functions you discussed in your previous calculus course, and in particular that

$$\sin^2 z + \cos^2 z = 1 \text{ for all } z \in \mathbf{R}.$$

## 12.4 The Exponential Function

**12.22 Example.** Suppose we had a complex function $E$ such that $E$ is everywhere differentable and

$$E' = E, \text{ and } E(0) = 1. \tag{12.23}$$

Let $H(z) = E(z)E(-z)$ for all $z \in \mathbf{C}$. By the chain and product rules,

$$H'(z) = E'(z)E(-z) + E(z)[-E'(-z)] = E(z)E(-z) - E(z)E(-z) = 0$$

on $\mathbf{C}$, so $H'$ is constant. Since $H(0) = E(0)E(0) = 1$, we conclude

$$E(z)E(-z) = 1 \text{ for all } z \in \mathbf{C}. \tag{12.24}$$

In particular $E(z)$ is never 0, and

$$E(-z) = (E(z))^{-1} \text{ for all } z \in \mathbf{C}.$$

Now let $a \in \mathbf{C}$ and define a function $H_a: \mathbf{C} \to \mathbf{C}$ by

$$H_a(z) = E(z+a)E(-z).$$

We have

$$\begin{aligned} H_a'(z) &= E'(z+a)E(-z) + E(z+a)[-E'(-z)] \\ &= E(z+a)E(-z) - E(z+a)E(-z) = 0 \end{aligned}$$

for all $z \in \mathbf{C}$, so $H_a$ is constant, and $H_a(0) = E(a)E(0) = E(a)$. Thus

$$E(z+a)E(-z) = E(a) \text{ for all } z \in \mathbf{C},\ a \in \mathbf{C},$$

and by (12.24),

$$E(z+a) = E(a)E(z) \text{ for all } z \in \mathbf{C},\ a \in \mathbf{C}. \tag{12.25}$$

Next suppose you know some function $e: \mathbf{R} \to \mathbf{R}$ such that $e'(t) = e(t)$ for all $t \in \mathbf{R}$ and $e(0) = 1$. (You do know such a function from your previous calculus course.) Let

$$K(t) = E(-t)e(t) \text{ for all } t \in \mathbf{R}.$$

Then by the product and chain rules,

$$K'(t) = [-E(-t)]e(t) + E(-t)e(t) = 0 \text{ for all } t \in \mathbf{R},$$

so $K$ is constant on $\mathbf{R}$, and since $K(0) = E(0)e(0) = 1$, we have $E(-t)e(t) = 1$. By (12.24),

$$e(t) = E(t) \text{ for all } t \in \mathbf{R}.$$

Now I will try to construct a function $E$ satisfying the differential equation (12.23) by hoping that $E$ is given by a power series. Suppose

$$\begin{aligned} E(z) &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 \cdots \text{ for all } z \in \mathbf{C}. \\ E(0) &= a_0 + 0 + 0 + \cdots. \end{aligned}$$

Since $E(0) = 1$, we must have $a_0 = 1$, and

$$E(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots.$$

By the differentiation theorem,

$$E'(z) = a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots,$$

and

$$a_1 = E'(0) = E(0) = 1.$$

By the differentiation theorem again,

$$E'(z) = 2 \cdot 1 a_2 + 3 \cdot 2 a_3 z + 4 \cdot 3 a_4 z^2 + \cdots,$$

so

$$2 \cdot 1 a_2 = E'(0) = E(0) = 1 \text{ and } a_2 = \frac{1}{2 \cdot 1}.$$

Hence

$$E(z) = E'(z) = 1 + 3 \cdot 2 a_3 z + 4 \cdot 3 a_4 z^2 + \cdots.$$

Repeating the process, we get

$$E'(z) = 3 \cdot 2 \cdot 1 a_3 + 4 \cdot 3 \cdot 2 a_4 z + \cdots,$$

so

$$3 \cdot 2 \cdot 1 a_3 = E'(0) = E(0) = 1 \text{ and } a_3 = \frac{1}{3 \cdot 2 \cdot 1}.$$

I see a pattern here: $a_n = \dfrac{1}{n!}$.

**12.26 Definition (Exponential function.)** Let $E$ denote the power series $\sum\left\{\frac{z^n}{n!}\right\} = \left\{1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}\right\}$. We will show in exercise 12.31 that $E$ has infinite radius of convergence. We write

$$E(z) = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ for all } z \in \mathbf{C}.$$

**12.27 Theorem.** $\exp' = \exp$ *and* $\exp(0) = 1$.

Proof: It is clear that $\exp(0) = 1$. The formal derivative of $E$ is

$$DE = \sum\left\{\frac{nz^{n-1}}{n!}\right\} = \sum\left\{\frac{(n+1)z^n}{(n+1)!}\right\} = \sum\left\{\frac{z^n}{n!}\right\} = E,$$

so the [still unproved] differentiation theorem says that $\exp' = \exp$. It follows from our discussion above that $\exp(z)$ is never 0,

$$\exp(-z) = (\exp(z))^{-1} \text{ for all } z \in \mathbf{C}, \tag{12.28}$$

and

$$\exp(a + z) = \exp(a)\exp(z) \text{ for all } z \in \mathbf{C}. \tag{12.29}$$

It is clear that $\exp(z)$ is real for all $z \in \mathbf{R}$. In fact, we must have $\exp(z) \in \mathbf{R}^+$ for all $z \in \mathbf{R}$, since exp is continuous (differentiable functions are continuous) and if $\exp(t) < 0$ for some $z$, the intermediate value theorem would say $\exp(y) = 0$ for some $y$ between 0 and $t$. Since $\exp'(t) = \exp(t) > 0$ on $\mathbf{R}$, exp is strictly increasing on $\mathbf{R}$. ‖

**12.30 Definition ($e$.)** We define $e$ to be the number $\exp(1)$; i.e., $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

**12.31 Exercise.** Show that $\sum\left\{\frac{z^n}{n!}\right\}$ has infinite radius of convergence.

**12.32 Exercise.** Use the definition of $e$ to show that $e > 2.718$.

**12.33 Exercise.**

a) Show that $\exp(nz) = (\exp(z))^n$ for all $n \in \mathbf{N}$, $z \in \mathbf{C}$.

b) Show that $\exp(nz) = (\exp(z))^n$ for all $n \in \mathbf{Z}$, $z \in \mathbf{C}$.

**12.34 Exercise.** From the previous exercise, it follows that

$$\exp(nz) = (\exp(z))^n \text{ for all } z \in \mathbf{C},\ n \in \mathbf{Z}.$$

Use this to prove that

$$\exp\left(\frac{p}{q}t\right) = (\exp(t))^{\frac{p}{q}} \text{ for all } t \in \mathbf{R},\ p \in \mathbf{Z},\ q \in \mathbf{Z}^+;$$

i.e.,

$$\exp(rt) = (\exp(t))^r \text{ for all } t \in \mathbf{R},\ r \in \mathbf{Q}.$$

(Note that for $t = 1$, this says

$$\exp(r) = (\exp(1))^r = e^r.$$

**12.35 Notation ($e^z$.)** Another notation for $\exp(z)$ is $e^z$. This notation is motivated by the previous exercise. With this notation, we have

$$\begin{aligned} e^{z+a} &= e^z e^a \text{ for all } z, a \in \mathbf{C}. \\ (e^z)^{-1} &= e^{-z} \text{ for all } z \in \mathbf{C}. \\ (e^t)^r &= e^{(tr)} \text{ for all } t \in \mathbf{R},\ r \in \mathbf{Q}. \end{aligned}$$

**12.36 Theorem.** *Every number $t \in \mathbf{R}^+$ can be written as $t = \exp(s)$ for a unique $s \in \mathbf{R}$.*

Proof: The uniqueness of $s$ follows from the fact that exp is strictly increasing on $\mathbf{R}$. Let $t \in (1, \infty)$. From the expansion $\exp(t) = 1 + t + \frac{t^2}{2!} + \cdots$, we see that $\exp(t) > t$. Since exp is continuous, we can apply the intermediate value theorem to exp on $[0, t]$ to conclude $t = \exp(s)$ for some $s \in (0, t)$. If $t \in (0, 1)$, then $\frac{1}{t} \in (1, \infty)$, so $\frac{1}{t} = e^s$ for some $s \in (0, \infty)$, and $t = e^{-s}$ where $-s \in (-\infty, 0)$. Since $1 = e^0$, the theorem has been proved in all cases. ‖

## 12.5 Logarithms

**12.37 Definition (Logarithm.)** Let $t \in \mathbf{R}^+$. The *logarithm of* $t$ is the unique number $s \in \mathbf{R}$ such that $e^s = t$. We denote the logarithm of $t$ by $\ln(t)$, Hence

$$e^{\ln(t)} = t \text{ for all } t \in \mathbf{R}^+. \tag{12.38}$$

**12.39 Remark.** Since $\ln(e^r)$ is the unique number $s$ such that $e^s = e^r$, it follows that

$$\ln(e^r) = r \text{ for all } r \in \mathbf{R}. \tag{12.40}$$

**12.41 Theorem.** *For all* $a, b \in \mathbf{R}^+$,

$$\ln(ab) = \ln a + \ln b.$$

Proof:

$$\begin{aligned} \ln(ab) &= \ln(e^{\ln a} \cdot e^{\ln b}) \quad (\text{by } (12.38)) \\ &= \ln(e^{(\ln a + \ln b)}) \\ &= \ln a + \ln b \quad (\text{by } (12.40)). \ \| \end{aligned}$$

**12.42 Exercise.** Show that

a) $\ln(a^{-1}) = -\ln(a)$ for all $a \in \mathbf{R}^+$.

b) $\ln(a^r) = r\ln(a)$ for all $a \in \mathbf{R}^+$, $r \in \mathbf{Q}$.

c) $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$ for all $a, b \in \mathbf{R}^+$.

**12.43 Remark.** It follows from the fact that exp is strictly increasing on $\mathbf{R}$ that ln is strictly increasing on $\mathbf{R}^+$: if $0 < t < s$, then both of the statements $\ln(t) = \ln(s)$ and $\ln(t) > \ln(s)$ lead to contradictions.

**12.44 Theorem (Continuity of** ln.**)** ln *is a continuous function on* $\mathbf{R}^+$.

Proof: Let $a \in \mathbf{R}^+$, and let $f$ be a sequence in $\mathbf{R}^+$ such that $f \to a$. I want to show that $\ln \circ f \to \ln(a)$. Let $N_{f-\tilde{a}}$ be a precision function for $f - \tilde{a}$. I want to construct a precision function $M$ for $\ln \circ f - \widetilde{\ln(a)}$.

Scratchwork: For all $\varepsilon \in \mathbf{R}^+$, and all $n \in \mathbf{N}$,

$$\begin{aligned} |\ln(f(n)) - \ln(a)| < \varepsilon &\iff \ln(a) - \varepsilon < \ln(f(n)) < \ln(a) + \varepsilon \\ &\iff e^{\ln(a)-\varepsilon} < f(n) < e^{\ln(a)+\varepsilon} \\ &\iff e^{\ln(a)-\varepsilon} - a < f(n) - a < e^{\ln(a)+\varepsilon} - a \end{aligned}$$

Note that since ln is strictly increasing, $e^{\ln(a)+\varepsilon} - a$ and $a - e^{\ln(a)-\varepsilon}$ are both positive. This calculation motivates the following definition:

For all $\varepsilon \in \mathbf{R}^+$, let

$$M(\varepsilon) = \max(N_{f-\tilde{a}}(e^{\ln(a)+\varepsilon} - a), N_{f-\tilde{a}}(a - e^{\ln(a)-\varepsilon})).$$

Then for all $n \in \mathbf{N}$, $\varepsilon \in \mathbf{R}^+$,

$$\begin{aligned} n \geq M(\varepsilon) &\implies \begin{cases} f(n) - a \leq |f(n) - a| \leq e^{\ln(a)+\varepsilon} - a \\ a - f(n) \leq |f(n) - a| \leq a - e^{\ln(a)-\varepsilon} \end{cases} \\ &\implies e^{\ln(a)-\varepsilon} < f(n) < e^{\ln(a)+\varepsilon} \\ &\implies \ln(a) - \varepsilon < \ln(f(n)) < \ln(a) + \varepsilon \\ &\implies |\ln(f(n)) - \ln(a)| < \varepsilon \end{aligned}$$

Hence $M$ is a precision function for $\ln \circ f - \widetilde{\ln(a)}$. ‖

**12.45 Theorem (Differentiability of ln.)** *The function* ln *is differentiable on* $\mathbf{R}^+$ *and*

$$\ln'(x) = \frac{1}{x} \text{ for all } x \in \mathbf{R}^+.$$

Proof: Let $a \in \mathbf{R}^+$ and let $\{x_n\}$ be a sequence in $\mathbf{R}^+ \backslash \{a\}$. Then

$$\frac{\ln(x_n) - \ln(a)}{x_n - a} = \frac{\ln(x_n) - \ln(a)}{e^{\ln(x_n)} - e^{\ln(a)}} = \frac{1}{\left(\frac{e^{\ln(x_n)} - e^{\ln(a)}}{\ln(x_n) - \ln(a)}\right)}.$$

(Note, I have not divided by 0.) Since ln is continuous, I know $\{\ln(x_n)\} \to \ln(a)$, and hence

$$\left\{\frac{e^{\ln(x_n)} - e^{\ln(a)}}{\ln(x_n) - \ln(a)}\right\} \to \exp'(\ln(a)) = e^{\ln(a)} = a.$$

Hence,

$$\lim \left\{\frac{\ln(x_n) - \ln(a)}{x_n - a}\right\} = \frac{1}{a};$$

i.e.,

$$\lim_{x \to a} \frac{\ln(x) - \ln(a)}{x - a} = \frac{1}{a}.$$

This shows that $\ln'(a) = \dfrac{1}{a}$.

## 12.6 Trigonometric Functions

Next we calculate $\exp(it)$ for $t \in \mathbf{R}$.

$$\exp(it) = \sum_{j=0}^{\infty} \frac{(it)^j}{j!}, \quad t \in \mathbf{R}.$$

Now $\{i^n\} = \{1, i, -1, -i, 1, i, -1, -i, \cdots\}$ and it is clear that $(i)^{2n} = (-1)^n \in \mathbf{R}$, $(i)^{2n+1} = i(-1)^n$ is pure imaginary. Hence,

$$\begin{aligned} \mathrm{Re}\,(\exp(it)) &= \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{(2j)!} = \cos t \\ \mathrm{Im}\,(\exp(it)) &= \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j+1}}{(2j+1)!} = \sin t; \end{aligned}$$

i.e.,

$$\exp(it) = \cos t + i \sin t \text{ for all } t \in \mathbf{R}. \tag{12.46}$$

For any complex number $(x, y) = x + iy$, we have

$$\begin{aligned} \exp(x + iy) &= \exp(x)\exp(iy) = \exp(x)[\cos(y) + i\sin(y)] \\ &= \exp(x)\cos(y) + i\exp(x)\sin(y). \end{aligned}$$

Since your calculator has buttons that calculate approximations to exp, sin and cos, you can approximately calculate the exponential of any complex number with a few key strokes.

The relation (12.46)

$$\exp(it) = \cos t + i \sin t$$

actually holds for all $t \in \mathbf{C}$, since

$$\begin{aligned} \sum_{j=0}^{n} \frac{(-1)^j z^{2j}}{(2j)!} + i \sum_{j=0}^{n} \frac{(-1)^j z^{2j+1}}{(2j+1)!} &= \sum_{j=0}^{n} \frac{(iz)^{2j}}{(2j)!} + \sum_{j=0}^{n} \frac{(i)^{2j+1} z^{2j+1}}{(2j+1)!} \\ &= \sum_{j=0}^{2n+1} \frac{(iz)^j}{j!}. \end{aligned}$$

Hence

$$e^{iz} = \cos z + i \sin z \text{ for all } z \in \mathbf{C}, \tag{12.47}$$

so

$$e^{-iz} = \cos z - i \sin z \text{ for all } z \in \mathbf{C}. \tag{12.48}$$

We can solve (12.47) and (12.48) for $\sin(z)$ and $\cos(z)$ to obtain

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \text{ for all } z \in \mathbf{C}. \tag{12.49}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \text{ for all } z \in \mathbf{C}. \tag{12.50}$$

From (12.47) it follows that

$$|e^{it}| = 1 \text{ for all } t \in \mathbf{R},$$

i.e., $e^{it}$ is in the unit circle for all $t \in \mathbf{R}$.

**12.51 Exercise (Addition laws for** sin **and** cos.**)** Prove that

$$\begin{aligned} \cos(z+a) &= \cos(z)\cos(a) - \sin(z)\sin(a) \\ \sin(z+a) &= \sin(z)\cos(a) + \cos(z)\sin(a) \end{aligned}$$

for all $z, a \in \mathbf{C}$.

By the addition laws, we have (for all $x, y \in \mathbf{C}$),

$$\cos(x+iy) = \cos(x)\cos(iy) - \sin(x)\sin(iy) \tag{12.52}$$

$$\sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy). \tag{12.53}$$

By (12.49) and (12.50)

$$\cos(iy) = \frac{e^{i(iy)} + e^{-i(iy)}}{2} = \frac{e^{y} + e^{-y}}{2}$$

and

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = i\left(\frac{e^{y} - e^{-y}}{2}\right).$$

**12.54 Definition (Hyperbolic functions.)** For all $z \in \mathbf{C}$, we define the *hyperbolic sine* and *hyperbolic cosine* of $z$ by

$$\begin{aligned}\sinh(z) &= \frac{e^z - e^{-z}}{2} \\ \cosh(z) &= \frac{e^z + e^{-z}}{2}.\end{aligned}$$

Note that if $z$ is real, $\sinh(z)$ and $\cosh(z)$ are real. Most calculators have buttons that calculate cosh and sinh. We can now rewrite (12.52) and (12.53) as

$$\begin{aligned}\cos(x+iy) &= \cos(x)\cosh(y) - i\sin(x)\sinh(y) \\ \sin(x+iy) &= \sin(x)\cosh(y) + i\cos(x)\sinh(y).\end{aligned}$$

These formulas hold true for all complex $x$ and $y$.

Since

$$\sin' = \cos, \qquad \cos' = -\sin, \qquad \sin(0) = 0 \qquad \text{and } \cos(0) = 1,$$

it follows from our discussion in example 10.45 that

$$\sin(x) \geq x - \frac{x^3}{6} \text{ and } \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

for all $x > 0$. In particular

$$\sin(x) \geq x\left(1 - \frac{x^2}{6}\right) > 0 \text{ for } 0 < x < \sqrt{6}$$

and

$$\cos(2) < 1 - \frac{4}{2} + \frac{16}{24} < 0.$$

Hence $\cos' = -\sin < 0$ on $(0,2)$, so cos is strictly decreasing on $[0,2]$. Moreover cos is continuous (since it is differentiable) so by the intermediate value theorem there is a number $c$ in $(0,2)$ such that $\cos(c) = 0$. Since cos is strictly decreasing on $(0,2)$ this number $c$ is unique. (Cf. exercise 5.48.)

**12.55 Definition ($\pi$.)** We define the real number $\pi$ by the condition $\frac{\pi}{2}$ is the unique number in $(0,2)$ satisfying $\cos\left(\frac{\pi}{2}\right) = 0$.

**12.56 Theorem.** exp *is periodic of period* $2\pi i$*; i.e.,*

$$\exp(z + 2\pi i) = \exp(z) \text{ for all } z \in \mathbf{C}.$$

Proof: Since $\sin^2 t + \cos^2 t = 1$ for all $t \in \mathbf{C}$, we have $\sin^2\left(\frac{\pi}{2}\right) = 1$, so $\sin\left(\frac{\pi}{2}\right) = \pm 1$. We have noted that $\sin t > 0$ on $(0, 2)$ so $\sin\left(\frac{\pi}{2}\right) = 1$. Hence

$$e^{\frac{i\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i,$$

and

$$e^{2i\pi} = \left(e^{\frac{i\pi}{2}}\right)^4 = i^4 = 1. \tag{12.57}$$

It follows that $e^{2\pi i+z} = e^{2\pi i}e^z = 1e^z = e^z$ for all $z \in \mathbf{C}$. ‖

**12.58 Entertainment.** If Maple or Mathematica are asked for the numerical values of $(-1)^{3.14}$ and $i^i$, they agree that

$$(-1)^{3.14} = -.9048\cdots - i \cdot .4257\cdots$$

and

$$i^i = .2078\cdots.$$

Can you propose a reasonable definition for $(-1)^z$ and $i^z$ when $z$ is an arbitrary complex number, that is consistent with these results? To be reasonable you would require that when $z \in \mathbf{Z}$, $(-1)^z$ and $i^z$ give the expected values, and

$$\begin{aligned} (-1)^{z+w} &= (-1)^z(-1)^w \text{ for all } z, w \in \mathbf{C}, \\ (i)^{z+w} &= i^z i^w \text{ for all } z, w \in \mathbf{C}. \end{aligned}$$

**12.59 Exercise.** Prove that:

a) $\cos\pi = -1$, and $\sin\pi = 0$.

b) $\cos\frac{3\pi}{2} = 0$, and $\sin\frac{3\pi}{2} = -1$.

c) $\cos 2\pi = 1$, and $\sin 2\pi = 0$.

d) $\sin(2\pi - t) = -\sin t$ for all $t \in \mathbf{C}$.

e) $\cos(2\pi - t) = \cos t$ for all $t \in \mathbf{C}$.

f) $\sin(\pi - t) = \sin t$ for all $t \in \mathbf{C}$.

g) $\cos(\pi - t) = -\cos t$ for all $t \in \mathbf{C}$.

h) $\sin(2\pi + t) = \sin t$ for all $t \in \mathbf{C}$.

i) $\cos(2\pi + t) = \cos t$ for all $t \in \mathbf{C}$.

**12.60 Theorem.** $\cos(2\pi) = 1$ *and* $\cos t < 1$ *for* $0 < t < 2\pi$.

Proof: From the previous exercise, $\cos(2\pi) = \cos(0) = 1$. We've noted that $\sin t > 0$ for $t \in \left(0, \frac{\pi}{2}\right]$,

$$\begin{aligned} t \in \left(\tfrac{\pi}{2}, \pi\right) &\implies \frac{\pi}{2} < t < \pi \implies 0 < \pi - t < \frac{\pi}{2} \\ &\implies \sin(\pi - t) > 0 \\ &\implies \sin(t) > 0. \end{aligned}$$

Hence $\sin(t) > 0$ for $t \in (0, \pi)$. Hence $\cos'(t) = -\sin(t) < 0$ for $t \in (0, \pi)$. Hence cos is strictly decreasing on $(0, \pi)$. Hence $\cos(x) < \cos(0) = 1$ for all $x \in (0, \pi)$.

Now

$$\begin{aligned} t \in (\pi, 2\pi) &\implies \pi < t < 2\pi \implies 0 < 2\pi - t < \pi \\ &\implies \cos(2\pi - t) < 1 \\ &\implies \cos t < 1, \end{aligned}$$

and since $\cos(\pi) = -1 < 1$, we've shown that $\cos t < 1$ for all $t \in (0, 2\pi)$. |||

**12.61 Theorem.** *Every point* $(x, y)$ *in the unit circle can be written as* $(x, y) = e^{it}$ *for a unique* $t \in [0, 2\pi)$.

Proof: We first show uniqueness.

Suppose $(x, y) = x + iy = e^{it} = e^{is}$ where $s, t \in [0, 2\pi)$. Without loss of generality, say $s \leq t$. Then

$$1 = \frac{e^{it}}{e^{is}} = e^{i(t-s)} = \cos(t - s) + i \sin(t - s),$$

and $t - s \in [0, 2\pi)$. By the previous theorem, 0 is the only number in $[0, 2\pi)$ whose cosine is 1, so $t - s = 0$, and hence $t = s$.

Let $(x, y)$ be a point in the unit circle, so $x^2+y^2 = 1$, and hence $-1 \le x \le 1$. Since $\cos(0) = 1$ and $\cos(\pi) = -1$, it follows from the intermediate value theorem that $x = \cos t$ for some $t \in [0, \pi]$. Then

$$y^2 = 1 - x^2 = 1 - \cos^2(t) = \sin^2(t),$$

so $y = \pm \sin(t)$.

$$\begin{aligned} y = \sin t &\implies (x, y) = (\cos t, \sin t) = e^{it} \\ y = -\sin t &\implies (x, y) = (\cos t, -\sin t) = (\cos(2\pi - t), \sin(2\pi - t)) = e^{i(2\pi - t)} \end{aligned}$$

and since $t \in [0, \pi]$, we have $2\pi - t \in [\pi, 2\pi]$. ‖

**12.62 Lemma.** *The set of all complex solutions to $e^z = 1$ is $\{2\pi in : n \in \mathbf{Z}\}$.*

Proof: By exercise 12.59

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + i0 = 1,$$

so

$$e^{2\pi in} = \left(e^{2\pi i}\right)^n = 1^n = 1.$$

Let $w = (a, b) = a + ib$ be any solution to $e^z = 1$; i.e.,

$$1 = e^{a+bi} = e^a e^{ib}.$$

By uniqueness of polar decomposition,

$$e^a = 1 \text{ and } e^{ib} = 1,$$

so $a = 0$ (since for real $a$, $e^a = 1 \iff a = 0$). We can write $\dfrac{b}{2\pi} = n + \varepsilon$ where $n \in \mathbf{Z}$ and $\varepsilon \in [0, 1)$ by theorem 5.30, so $b = 2\pi n + 2\pi\varepsilon$ where $2\pi\varepsilon \in [0, 2\pi)$. Now

$$1 = e^{ib} = e^{2\pi in + i2\pi\varepsilon} = e^{2\pi i\varepsilon}.$$

By theorem 12.61, $2\pi i\varepsilon = 0$, so $\varepsilon = 0$, and $b = 2\pi n$; i.e., $w = 2\pi in$. ‖

**12.63 Definition (Argument.)** Let $a \in \mathbf{C}\backslash\{0\}$ and write $a$ in its polar decomposition $a = |a|u$, where $|u| = 1$. We know $u = e^{iA}$ for a unique $A \in [0, 2\pi)$. I will call $A$ the *argument* of $a$ and write $A = \mathrm{Arg}(a)$. Hence

$$a = |a|e^{i\mathrm{Arg}(a)} \quad A \in [0, 2\pi).$$

**12.64 Remark.** Our definition of Arg is rather arbitrary. Other natural definitions are

$$\text{Arg}_1(z) \text{ is the unique number } a \text{ in } [-\pi, \pi) \text{ such that } z = |z|e^{ia},$$

or

$$\text{Arg}_2(z) \text{ is the unique number } b \text{ in } (-\pi, \pi] \text{ such that } z = |z|e^{ib}.$$

None of these argument functions is continuous; e.g.,

$$\left\{e^{\frac{-i\pi}{n}}\right\}_{n\geq 1} \to 1.$$

But

$$\left\{\text{Arg}\left(e^{\frac{-i\pi}{n}}\right)\right\}_{n\geq 1} = \left\{\left(2\pi - \frac{\pi}{n}\right)\right\}_{n\geq 1} \to 2\pi \neq \text{Arg}(1).$$

**12.65 Theorem.** *Let $a \in \mathbf{C}\backslash\{0\}$. Then the complex solutions to the equation $e^z = a$ are exactly the numbers of the form*

$$z = \ln|a| + i\text{Arg}(a) + 2\pi i n \textit{ where } n \in \mathbf{Z}.$$

*In particular, every non-zero $a \in \mathbf{C}$ is the exponential of some $z \in \mathbf{C}$.*

Proof: Since

$$\begin{aligned} e^{(\ln|a|+i\text{Arg}(a)+2\pi i n)} &= e^{\ln|a|}e^{i\text{Arg}(a)}e^{2\pi i n} \\ &= |a|e^{i\text{Arg}(a)} = a, \end{aligned}$$

the numbers given are solutions to $e^z = a$. Let $w$ be any solution to $e^w = a$. Then $e^{w-\ln|a|-i\text{Arg}(a)} = \frac{a}{a} = 1$. Hence, by the lemma 12.62,

$$w - \ln|a| + i\text{Arg}(w) = 2\pi i n \text{ for some } n \in \mathbf{Z}. \;\|$$

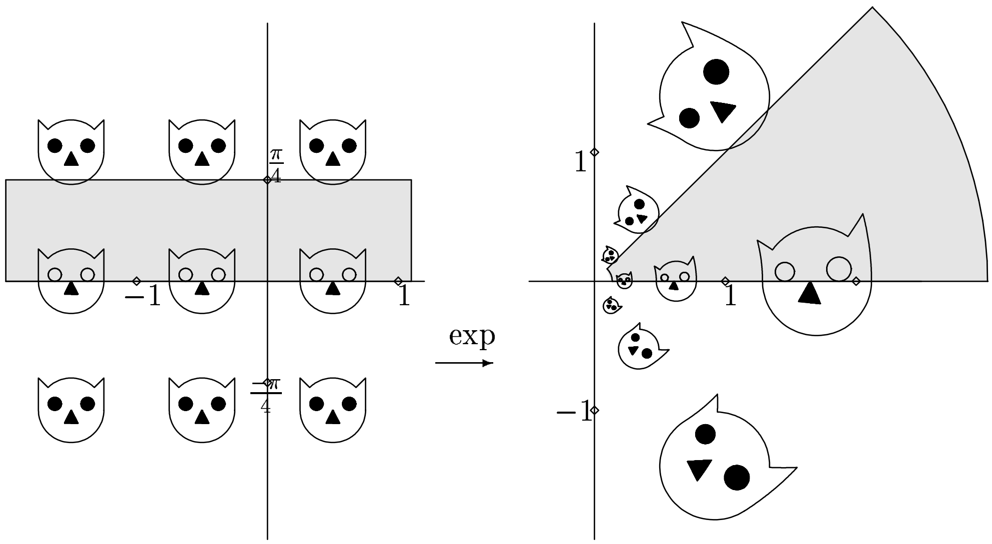
We will now look at exp geometrically as a function from $\mathbf{C}$ to $\mathbf{C}$.
Claim: exp maps the vertical line $x = x_0$ into the circle $C(0, e^{x_0})$.
Proof: If $z = x_0 + iy$, then

$$|e^z| = |e^{x_0+iy}| = |e^{x_0}e^{iy}| = |e^{x_0}||e^{iy}| = e^{x_0}.$$

Claim: exp maps the horizontal line $y = y_0$ into the ray through 0 with direction $e^{iy_0}$.
Proof: If $z = x + iy_0$, then

$$e^z = e^{x+iy_0} = e^x \cdot e^{iy_0} \text{ and } e^x > 0.$$

Since exp is periodic of period $2\pi i$, exp maps an infinite horizontal strip of width $w$ into an infinite circular segment making "angle $w$" at the origin.

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exp maps every strip $\{(x,y)\colon y_0 \le y < y_0 + 2\pi\}$ onto all of $\mathbf{C}\backslash\{0\}$.

**12.66 Theorem (Roots of complex numbers.)** *Let $a \in \mathbf{C}\backslash\{0\}$ and let $n \in \mathbf{Z}^+$. Then the solutions to $z^n = a$ in $\mathbf{C}$ are exactly the numbers*

$$z = |a|^{\frac{1}{n}} e^{i\left(\frac{\mathrm{Arg}(a)+2\pi k}{n}\right)} \text{ where } k \in \mathbf{Z} \text{ and } 0 \le k < n.$$

*(These numbers are distinct.)*

Proof: These numbers are clearly solutions to $z^n = a$. Let $w = |w|e^{i\mathrm{Arg}(w)}$ be any solution to $z^n = a$. Then

$$|w|^n e^{in\mathrm{Arg}(w)} = w^n = a = |a|e^{i\mathrm{Arg}(a)}.$$

By uniqueness of polar decomposition,

$$|w|^n = |a| \text{ and } e^{in\mathrm{Arg}(w)} = e^{i\mathrm{Arg}(a)},$$

i.e., $|w| = |a|^{1/n}$ and $e^{i[n\mathrm{Arg}(w)-\mathrm{Arg}(a)]} = 1$. Hence, $n\mathrm{Arg}(w) - \mathrm{Arg}(a) = 2\pi k$ for some $k \in \mathbf{N}$ and

$$\mathrm{Arg}(w) = \frac{\mathrm{Arg}(a) + 2\pi k}{n} \text{ for some } k \in \mathbf{N}.$$

Thus

$$e^{i\mathrm{Arg}w} = e^{i\left(\frac{\mathrm{Arg}(a)+2\pi k}{n}\right)} \text{ for some } k \in \mathbf{N}.$$

For each $k \in \mathbf{Z}$, the number

$$w_k = |w|^{\frac{1}{n}} e^{\frac{i\mathrm{Arg}(a)}{n}} \cdot e^{\frac{2\pi ik}{n}}$$

is a solution to $w^n = a$. For $0 \leq k < n$, the numbers $\dfrac{2\pi ik}{n}$ are distinct numbers in $[0, 2\pi)$, so the numbers $e^{\frac{2\pi ik}{n}}$ are distinct. For every $K \in \mathbf{Z}$, $\dfrac{K}{n} = M + \varepsilon$ where $M \in \mathbf{Z}$ and $\varepsilon \in [0, 1)$, so $K = nM + \varepsilon n$ where $\varepsilon n \in [0, n)$ and $\varepsilon n = K - nM \in \mathbf{Z}$; i.e.,

$$K = nM + k \text{ where } k \in \mathbf{Z} \text{ and } 0 \leq k < n.$$

Then $\frac{K}{n} = M + \frac{k}{n}$, so

$$e^{2\pi i\frac{K}{n}} = e^{2\pi iM} e^{\frac{2\pi ik}{n}} = e^{\frac{2\pi ik}{n}} \text{ where } k \in \mathbf{Z} \text{ and } 0 \leq k < n. \quad \|$$

## 12.7 Special Values of Trigonometric Functions

We have

$$\cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \cos\frac{\pi}{2}\cos\frac{\pi}{4} + \sin\frac{\pi}{2}\sin\frac{\pi}{4} = \sin\frac{\pi}{4}.$$

Hence $1 = \cos^2\left(\frac{\pi}{4}\right) + \sin^2\left(\frac{\pi}{4}\right) = 2\sin^2\left(\frac{\pi}{4}\right)$, and hence $\left(\cos\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$. Since we know sin is positive on $(0, \pi)$, we conclude that

$$\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

Observe that if $t \in \mathbf{R}$, then the problem of calculating $\cos(t)$ and $\sin(t)$ is the same as the problem of calculating $e^{it}$. Let $n \in \mathbf{Z}^+$. We know that the complex solutions of $z^n - 1 = 0$ are

$$\left\{e^{\frac{2\pi ik}{n}} : 0 \leq k < n,\ k \in \mathbf{Z}\right\},$$

so if we can express the solutions to $z^n - 1 = 0$ in algebraic terms, then we can express $\sin\left(\frac{2\pi k}{n}\right)$ and $\cos\left(\frac{2\pi k}{n}\right)$ in algebraic terms. We have

$$z^6-1=0 \iff (z^3-1)(z^3+1)=0 \iff (z-1)(z^2+z+1)(z+1)(z^2-z+1)=0.$$

Here $z = 1$ and $z = -1$ are obvious sixth roots of 1, and the other four roots are the solutions of the quadratic equations

$$z^2 + z + 1 = 0 \text{ and } z^2 - z + 1 = 0.$$

**12.67 Exercise.** Find the solutions to $z^2 + z + 1 = 0$ and $z^2 - z + 1 = 0$ in terms of square roots of rational numbers. These solutions are

$$\left\{e^{\frac{\pi i}{3}}, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}, e^{\frac{5\pi i}{3}}\right\}.$$

Identify each solution with one of these exponentials. Find $\cos\left(\frac{\pi}{3}\right)$ and $\sin\left(\frac{\pi}{3}\right)$.

**12.68 Exercise.** Use the fact that

$$e^{\frac{\pi i}{6}} = e^{\frac{\pi i}{2}} \cdot e^{-\frac{\pi i}{3}}$$

to find $\cos\frac{\pi}{6}$ and $\sin\frac{\pi}{6}$. ‖

The numbers $\cos\left(\frac{2\pi}{5}\right)$ and $\sin\left(\frac{2\pi}{5}\right)$ can also be expressed algebraically. If $z = e^{\frac{2\pi i}{5}}$, then $z^5 - 1 = 0$, so

$$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$$

and since $z \neq 1$,

$$(z^4 + z^3 + z^2 + z + 1) = 0.$$

The fact that $z^5 = 1$ says $z^{-1} = z^4$ and $z^{-2} = z^3$, so

$$1 + z + z^{-1} + z^2 + z^{-2} = 0;$$

i.e.,

$$1+e^{\frac{2\pi i}{5}}+e^{\frac{-2\pi i}{5}}+e^{\frac{4\pi i}{5}}+e^{\frac{-4\pi i}{5}}=0,$$

or

$$1+2\cos\left(\frac{2\pi}{5}\right)+2\cos\left(\frac{4\pi}{5}\right)=0.$$

Now for all $z\in\mathbf{C}$,

$$\cos(2z)=\cos(z+z)=\cos^2(z)-\sin^2(z)=\cos^2(z)-\Big(1-\cos^2(z)\Big)=2\cos^2(z)-1,$$

so

$$1+2\cos\left(\frac{2\pi}{5}\right)+2\left(2\cos^2\left(\frac{2\pi}{5}\right)-1\right)=0. \tag{12.69}$$

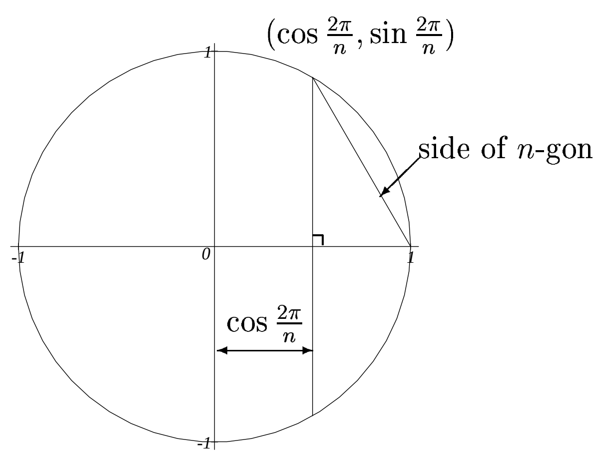
Hence $\cos\left(\frac{2\pi}{5}\right)$ satisfies a quadratic equation.

**12.70 Exercise.**

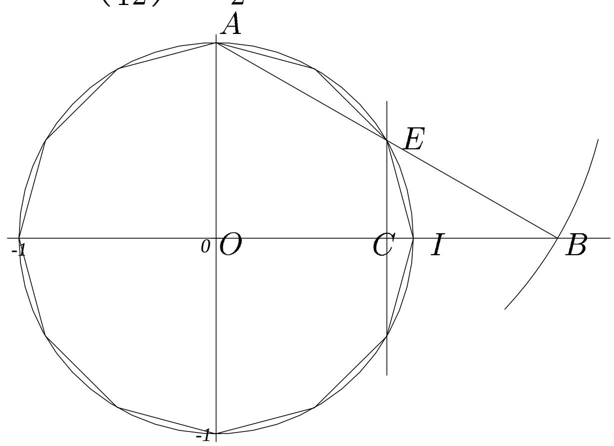
a) Solve (12.69), and determine $\cos\left(\frac{2\pi}{5}\right)$ and $\sin\left(\frac{2\pi}{5}\right)$ in algebraic terms.

b) The quadratic equation has two solutions, one of which is $\cos\left(\frac{2\pi}{5}\right)$. What is the geometrical significance of the other solution?

**12.71 Entertainment.** The algebraic representation for $\cos\left(\frac{2\pi}{5}\right)$ shows that a regular pentagon can be inscribed in a given circle. Let a circle be given, and call its radius 1. If you can construct $\cos\left(\frac{2\pi}{n}\right)$ with compass and straightedge (see the figure), then you can construct a side of a regular $n$-gon inscribed in the circle (and hence you can construct the $n$-gon).



For example, since $\cos\left(\frac{2\pi}{12}\right) = \frac{\sqrt{3}}{2}$, we can construct a dodecagon as follows:




Construction of a Dodecagon

In the figure, make an arc of radius 2 with center at $A$, intersecting the $x$-axis at $B$. Then $OB = \sqrt{3}$, so if $C$ bisects $OB$, then $OC = \cos\left(\frac{2\pi}{12}\right)$, and the vertical line through $C$ intersects the circle at $E$ where $IE$ is a side of the 12-gon.

Use the formula for $\cos\left(\frac{2\pi}{5}\right)$ to inscribe a regular pentagon in a circle.

**12.72 Entertainment.** (This problem entertained Gauss. It will probably not really entertain you, unless you are another Gauss.) Show that a regular 17-gon can be inscribed in a circle using compasses and straightedge.

Gauss discovered this result in 1796 [31, p 754] when he was a nineteen year old student at Göttingen. The result is [21, p 458]

$$\begin{aligned}\cos\left(\frac{2\pi}{17}\right) &= -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} \\ &\quad + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{(34 - 2\sqrt{17})} - 2\sqrt{34 + 2\sqrt{17}}}.\end{aligned}$$

## 12.8 Proof of the Differentiation Theorem

**12.73 Lemma.** *The power series* $\sum\{nz^n\}$ *has radius of convergence equal to* 1.

**12.74 Exercise.** Prove lemma 12.73. (We proved this lemma earlier using the differentiation theorem. Since we need this result to prove the differentiation theorem, we now want a proof that does *not* use the differentiation theorem.)

**12.75 Lemma.** *Let $\sum\{a_n z^n\}$ be a power series. Then the two series $\sum\{a_n z^n\}$ and $\sum\{na_n z^{n-1}\}$ have the same radius of convergence.*

Proof: I'll show that for all $w, v \in \mathbf{C}\backslash\{0\}$.

a) If $\sum\{|na_n w^{n-1}|\}$ converges, then $\{\sum |a_n w^n|\}$ converges.

b) If $\sum\{|a_n w^n|\}$ converges and $|v| < |w|$, then $\sum\{|na_n v^{n-1}|\}$ converges.

a) follows from the comparison test, since

$$|a_n w^n| \leq |na_n w^{n-1}| \cdot |w| \text{ for all } n \in \mathbf{Z}^+.$$

To prove b), suppose $\sum\{|a_n w^n|\}$ converges and $|v| < |w|$. By lemma 12.73, $\sum\left\{n\left|\frac{v}{w}\right|^n\right\}$ converges, and hence $\left\{n\left|\frac{v}{w}\right|^n\right\}$ is bounded. Choose $M \in \mathbf{R}^+$ such that

$$n\left|\frac{v}{w}\right|^n \leq M \text{ for all } n \in \mathbf{N}.$$

Then $n|v|^n < M|w^n|$, and

$$\left|a_n n|v|^{n-1}\right| \leq |a_n w^n| \cdot \frac{M}{|v|} \text{ for all } n \in \mathbf{N}.$$

By the comparison test, $\sum\{|a_n n v^{n-1}|\}$ converges. |||

**12.76 Corollary.** *$\sum\{a_n z^n\}$ and $\sum\{a_n n(n-1)z^{n-2}\}$ have the same radius of convergence.*

Proof: Use the lemma twice. |||

**12.77 Theorem.** *Let $\sum\{c_n z^n\}$ be a power series with positive radius of convergence. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z$ in the disc of convergence for $f$ and let $Df(z) = \sum_{n=1}^{\infty} nc_n z^{n-1}$ be the function corresponding to the formal derivative of $\sum\{c_n z^n\}$. Then $f$ is differentiable on its disc of convergence, and $f'(a) = Df(a)$ for all $a$ in the disc of convergence.*

Proof: Let $a$ be a point in the disc of convergence, and let $z$ be a different point in the disc. Then

$$\begin{aligned} f(z) - f(a) &= \sum_{n=0}^{\infty} c_n z^n - \sum_{n=0}^{\infty} c_n a^n \\ &= \sum_{n=1}^{\infty} c_n (z^n - a^n) \qquad \text{(since } z^0 = a^0\text{)} \\ &= \sum_{n=1}^{\infty} c_n (z-a) \sum_{j=0}^{n-1} z^{n-1-j} a^j \\ &= (z-a) \sum_{n=1}^{\infty} c_n \sum_{j=0}^{n-1} z^{n-1-j} a^j. \end{aligned}$$

Let

$$D_a f(z) = \sum_{n=1}^{\infty} c_n \sum_{j=0}^{n-1} z^{n-1-j} a^j.$$

Then

$$f(z) - f(a) = (z-a) D_a f(z),$$

and since

$$D_a f(a) = \sum_{n=1}^{\infty} c_n \sum_{j=0}^{n-1} a^{n-1} = \sum_{j=1}^{\infty} c_n n a^{n-1} = Df(a),$$

the theorem will follow if we can show that $D_a f$ is continuous at $a$.

In the calculation below, I quietly use the following facts:

a) When $n = 1$, $\sum_{j=0}^{n-1} z^{n-1-j} a^j - \sum_{j=0}^{n-1} a^{n-1-j} a^j = 0$.

b) When $j = n-1$, $z^{n-1-j} - a^{n-1-j} = 0$.

$$\begin{aligned} D_a f(z) - D_a f(a) &= \sum_{n=1}^{\infty} c_n \sum_{j=0}^{n-1} z^{n-1-j} a^j - \sum_{n=1}^{\infty} c_n \sum_{j=0}^{n-1} a^{n-1-j} a^j \\ &= \sum_{n=2}^{\infty} c_n \sum_{j=0}^{n-1} a^j (z^{n-1-j} - a^{n-1-j}) \\ &= \sum_{n=2}^{\infty} c_n \sum_{j=0}^{n-2} a^j (z-a) \sum_{k=0}^{n-2-j} z^{n-2-j-k} a^k \\ &= (z-a) \sum_{n=2}^{\infty} c_n \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} z^{n-2-j-k} a^{j+k}. \qquad (12.78) \end{aligned}$$

Let the radius of convergence of our power series be $R$, and let $\varepsilon = \dfrac{R - |a|}{2}$. Then

$$\begin{aligned} |z-a| < \varepsilon &\implies |z| - |a| \le |z-a| < \varepsilon \\ &\implies |z| < |a| + \varepsilon = |a| + \frac{R-|a|}{2} = \frac{R+|a|}{2}. \end{aligned}$$

Let $S = \dfrac{R+|a|}{2} < R$. Then $|a| < S$, and

$$\begin{aligned} |z-a| < \varepsilon &\implies |z| < S \\ &\implies \left| \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} z^{n-2-j-k} a^{j+k} \right| \le \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} |z|^{n-2-j-k} |a|^{j+k} \\ &\le \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} S^{n-2} = S^{n-2} \sum_{j=0}^{n-2} (n-1-j) \\ &\le S^{n-2} \sum_{j=0}^{n-2} n \le S^{n-2} \sum_{j=0}^{n-1} n \\ &= S^{n-2} \cdot \frac{n(n-1)}{2} \le S^{n-2} \cdot n(n-1). \end{aligned}$$

(Here I've used the fact that $n - 1 - j \le n$ for $0 \le j \le n-2$.) Thus

$$|z-a| < \varepsilon \implies \sum_{n=2}^{\infty} \left| c_n \sum_{j=0}^{n-2} \sum_{k=0}^{n-2-j} z^{n-2-j-k} a^{j+k} \right| \le \sum_{n=2}^{\infty} |c_n| S^{n-2} \cdot n(n-1).$$

We noticed in the corollary to lemma 12.75 that the series $\sum\{n(n-1)c_n z^{n-2}\}$ has radius of convergence $R$, and hence $\sum\{|c_n|S^{n-2}n(n-1)\}_{n\ge 2}$ converges to a limit $M$, and by (12.78),

$$|D_a f(z) - D_a f(a)| \le |z-a| \cdot M \text{ whenever } |z-a| < \varepsilon.$$

If $\{w_n\}$ is a sequence in $\operatorname{dom}(D_a f)$ such that $\{w_n\} \to a$, then

$$|D_a f(w_n) - D_a f(a)| \le |w_n - a| \cdot M$$

for all large $n$, and by the null-times bounded theorem and comparison theorem for null sequences, $\{D_a f(w_n)\} \to D_a f(a)$. Hence, $D_a f$ is continuous at $a$. ‖

## 12.9 Some XVIII-th Century Calculations

The following proofs that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c = \frac{\pi^2}{6}$$

and

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c = \frac{\pi^2}{8}$$

use XVIII-th century standards or rigor. You should decide what parts are justified. I denote $f'(\theta)$ by $\dfrac{df}{d\theta}$ below. By the geometric series formula,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

If $z = re^{i\theta}$ where $r > 0$, $\theta \in \mathbf{R}$, then

$$\sum_{n=0}^{\infty} r^n e^{in\theta} = \frac{1}{1 - re^{i\theta}} \cdot \frac{1 - re^{-i\theta}}{1 - re^{-i\theta}}$$

so

$$\sum_{n=0}^{\infty} \left(r^n \cos(n\theta) + ir^n \sin(n\theta)\right) = \frac{1 - re^{-i\theta}}{1 - r(e^{i\theta} + e^{-i\theta}) + r^2} = \frac{(1 - r\cos\theta) + ir\sin\theta}{1 + r^2 - 2r\cos\theta}.$$

By equating the real and imaginary parts, we get

$$\sum_{n=0}^{\infty} r^n \cos n\theta = \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}, \quad \sum_{n=0}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 + r^2 - 2r\cos\theta}.$$

For $r = 1$, this yields

$$\sum_{n=0}^{\infty} \cos n\theta = \frac{1 - \cos\theta}{2 - 2\cos\theta} = \frac{1}{2}.$$

Thus, $1 + \sum_{n=1}^{\infty} \cos n\theta = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \cos n\theta = -\frac{1}{2}.$$

Hence,

$$\frac{d}{d\theta}\left(\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}\right)=\frac{d}{d\theta}\left(-\frac{1}{2}\theta\right).$$

Since two antiderivatives of a function differ by a constant

$$\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}=-\frac{1}{2}\theta+C$$

for some constant $C$. When $\theta=\pi$, we get

$$0=\sum_{n=1}^{\infty}\frac{\sin n\pi}{n}=-\frac{1}{2}\pi+C$$

so $C=\frac{1}{2}\pi$ and thus

$$\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}=\frac{1}{2}(\pi-\theta). \tag{12.79}$$

For $\theta=\frac{\pi}{2}$, this gives us

$$\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\&c=\frac{1}{2}\left(\pi-\frac{\pi}{2}\right)=\frac{\pi}{4}$$

(which is the Gregory-Leibniz-Madhava formula). We can rewrite (12.79) as

$$\frac{d}{d\theta}\sum_{n=1}^{\infty}-\frac{\cos n\theta}{n^2}=\frac{d}{d\theta}\left(-\frac{(\pi-\theta)^2}{4}\right).$$

Again, since two antiderivatives of a function differ by a constant, there is a constant $C_1$ such that

$$\sum_{n=1}^{\infty}\frac{\cos n\theta}{n^2}=\frac{(\pi-\theta)^2}{4}+C_1.$$

For $\theta=0$, this says

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{4}+C_1,$$

and for $\theta=\pi$, this says

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=C_1.$$

Subtract the second equation from the first to get

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = \frac{\pi^2}{4};$$

i.e.,

$$2 + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \&c = \frac{\pi^2}{4},$$

and thus

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c = \frac{\pi^2}{8}. \qquad (12.80)$$

Let $S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c$. Subtract (12.80) from this to get

$$\begin{aligned} S - \frac{\pi^2}{8} &= \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \&c = \frac{1}{4 \cdot 1} + \frac{1}{4 \cdot 2^2} + \frac{1}{4 \cdot 3^2} + \&c \\ &= \frac{1}{4}\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \&c\right] = \frac{1}{4}S. \end{aligned}$$

Hence, $\frac{3}{4}S = \frac{\pi^2}{8}$, and then $S = \frac{\pi^2}{6}$. |||

An argument similar to the following was given by Jacob Bernoulli in 1689 [31, p 443]. Let

$$N = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \&c.$$

Then

$$N - 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \&c.$$

Subtract the second series from the first to get

$$\begin{aligned} 1 &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \&c \\ &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \&c. \end{aligned}$$

Therefore,

$$1 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \&c.$$

**12.81 Exercise.**

a) Explain why Bernoulli's argument is not valid.

b) Give a valid argument proving that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

**12.82 Note.** The notation $\pi$ was introduced by William Jones in 1706 to represent the ratio of the circumference to the diameter of a circle[15, vol2, p9]. Both Maple and Mathematica designate $\pi$ by `Pi` .

The notation $e$ was introduced by Euler in 1727 or 1728 to denote the base of natural logarithms[15, vol 2, p 13]. In Mathematica $e$ is denoted by `E` . In the current version of Maple there is no special name for $e$; it is denoted by `exp(1)` .