# Chapter 9

# Properties of Continuous Functions

## 9.1 Extreme Values

**9.1 Definition (Maximum, Minimum.)** Let $f: S \to \mathbf{R}$ be a function from a set $S$ to $\mathbf{R}$, and let $a \in S$. We say that $f$ *has a maximum at* $a$ if $f(a) \geq f(x)$ for all $x \in S$, and we say $f$ *has a minimum at* $a$ if $f(a) \leq f(x)$ for all $x \in S$.

**9.2 Definition (Maximizing set.)** Let $f: S \to \mathbf{R}$ be a function and let $M$ be a subset of $S$. We say $M$ is a *maximizing set* for $f$ on $S$ if for each $x \in S$ there is a point $m \in M$ such that $f(m) \geq f(x)$.

**9.3 Examples.** If $f$ has a maximum at $a$ then $\{a\}$ is a maximizing set for $f$ on $S$.

If $M$ is a maximizing set for $f$ on $S$, and $M \subset B \subset S$, then $B$ is also a maximizing set for $f$ on $S$.

If $f: S \to \mathbf{R}$ is any function (with $S \neq \emptyset$), then $S$ is a maximizing set for $f$ on $S$, so every function with non-empty domain has a maximizing set.

Let

$$f(z) = \begin{cases} \frac{1}{|z|} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0. \end{cases}$$

Then every disc $D(0,\varepsilon)$ is a maximizing set for $f$, since if $z \in \mathbf{C}\backslash\{0\}$ we can find $n \in \mathbf{N}$ with $n > \max\left(\frac{1}{\varepsilon}, \frac{1}{|z|}\right)$; then $n > \frac{1}{\varepsilon}$, so $\frac{1}{n} < \varepsilon$, so $\frac{1}{n} \in D(0,\varepsilon)$ and $f\left(\frac{1}{n}\right) = n > \frac{1}{|z|} = f(z)$. This argument shows that $\left\{\frac{1}{n+1} : n \in \mathbf{N}\right\}$ is also a maximizing set for $f$.

**9.4 Remark.** Let $S$ be a set, and let $f: S \to \mathbf{R}$, and let $M$ be a subset of $S$. If $M$ is *not* a maximizing set for $f$ on $S$, then there is some point $x \in S$ such that $f(x) > f(m)$ for all $m \in M$.

**9.5 Lemma.** *Let $S$ be a set, let $f: S \to \mathbf{R}$ be a function, and let $M$ be a maximizing set for $f$ on $S$. If $M = A \cup B$, then at least one of $A, B$ is a maximizing set for $f$ on $S$.*

Proof: Suppose $A \cup B$ is a maximizing set for $f$ on $S$, but $A$ is not a maximizing set for $f$ on $S$. Then there is some $s \in S$ such that for all $a \in A$, $f(s) > f(a)$. Since $A \cup B$ is a maximizing set for $f$ on $S$, there is an element $t$ in $A \cup B$ such that $f(t) \geq f(s)$, so $f(t) > f(a)$ for all $a \in A$, so $t \notin A$, so $t \in B$. Now, for every $x \in S$ there is an element $c$ in $A \cup B$ with $f(c) \geq f(x)$. If $c \in A$, then the element $t \in B$ satisfies $f(t) > f(c) \geq f(x)$ so there is some element $u \in B$ with $f(u) \geq f(x)$ (if $c \in A$, take $u = t$; if $c \in B$, take $u = c$.) Hence $B$ is a maximizing set for $f$ on $S$. |||

**9.6 Theorem (Extreme value theorem.)** *Let $a, b \in \mathbf{R}$ with $a < b$ and let $f: [a,b] \to \mathbf{R}$ be a continuous function. Then $f$ has a maximum and a minimum on $[a,b]$.*

Proof: We will construct a binary search sequence $\{[a_n, b_n]\}$ with $[a_0, b_0] = [a,b]$ such that each interval $[a_n, b_n]$ is a maximizing set for $f$ on $[a,b]$. We put

$$\begin{aligned} [a_0, b_0] &= [a,b] \\ [a_{n+1}, b_{n+1}] &= \begin{cases} \left[a_n, \frac{a_n+b_n}{2}\right] & \text{if } \left[a_n, \frac{a_n+b_n}{2}\right] \text{ is a maximizing set for } f \\ \left[\frac{a_n+b_n}{2}, b_n\right] & \text{otherwise.} \end{cases} \end{aligned}$$

By the preceding lemma (and induction), we see that each interval $[a_n, b_n]$ is a maximizing set for $f$ on $[a,b]$. Let $c$ be the number such that $\{[a_n, b_n]\} \to c$

and let $s \in [a,b]$. Since $[a_n, b_n]$ is a maximizing set for $f$ on $[a,b]$, there is a number $s_n \in [a_n, b_n]$ with $f(s_n) \geq f(s)$. Since

$$a_n \leq c \leq b_n \text{ and } a_n \leq s_n \leq b_n,$$

we have $|s_n - c| \leq |b_n - a_n| = \dfrac{(b-a)}{2^n}$, so $\{s_n\} \to c$. By continuity of $f$, $\{f(s_n)\} \to f(c)$. Since $f(s_n) \geq f(s)$, it follows by the inequality theorem for limits that

$$f(c) = \lim\{f(s_n)\} \geq f(s).$$

Hence $c$ is a maximum point for $f$ on $[a,b]$. This shows that $f$ has a maximum. Since $-f$ is also a continuous function on $[a,b]$, $-f$ has a maximum on $[a,b]$; i.e., there is a point $p \in [a,b]$ such that $-f(p) \geq -f(x)$ for all $x \in [a,b]$. Then $f(p) \leq f(x)$ for all $x \in [a,b]$, so $f$ has a minimum at $p$. |||

**9.7 Definition (Upper bound.)** Let $S$ be a subset of $\mathbf{R}$, let $b, B \in \mathbf{R}$. We say $B$ is an *upper bound* for $S$ if $x \leq B$ for all $x \in S$, and we say $b$ is a *lower bound* for $S$ if $b \leq x$ for all $x \in S$.

**9.8 Remark.** If $S$ is a bounded subset of $R$ and $B$ is a bound for $S$, then $B$ is an upper bound for $S$ and $-B$ is a lower bound for $S$, since

$$|x| \leq B \implies -B \leq x \leq B.$$

Conversely, if a subset $S$ of $\mathbf{R}$ has an upper bound $B$ and a lower bound $b$,then $S$ is bounded, and $\max(|b|, |B|)$ is a bound for $S$, since

$$b \leq x \leq B \implies -\max(|b|,|B|) \leq -|b| \leq b \leq x \leq B \leq |B| \leq \max(|b|,|B|).$$

**9.9 Theorem (Boundedness theorem.)** *Let $a, b \in \mathbf{R}$ with $a < b$ and let $f\colon [a,b] \to \mathbf{R}$ be a continuous function. Then $f$ is bounded on $[a,b]$.*

Proof: By the extreme value theorem, there are points $p, q \in [a,b]$ such that

$$f(p) \leq f(x) \leq f(q) \text{ for all } x \in [a,b].$$

Hence $f([a,b])$ has an upper bound and a lower bound, so $f([a,b])$ is bounded. |||

**9.10 Exercise.** Give examples of the functions described below, or explain why no such function exists. Describe your functions by formulas if you can, but pictures of graphs will do if a formula seems too complicated.

a) $f\colon [0,1] \to \mathbf{R}$, $f$ is not bounded.

b) $g\colon (0,1) \to \mathbf{R}$, $g$ is continuous, $g$ is not bounded.

c) $h\colon [0,\infty) \to \mathbf{R}$, $h$ is continuous, $h$ is not bounded.

d) $k\colon [0,\infty) \to \mathbf{R}$, $k$ is strictly increasing, $k$ is continuous, $k$ is bounded.

e) $l\colon [0,1] \to \mathbf{R}$, $l$ is continuous, $l$ is not bounded.

## 9.2 Intermediate Value Theorem

**9.11 Theorem (Intermediate Value Theorem.)** *Let $a, b \in \mathbf{R}$ with $a < b$, and let $f\colon [a,b] \to \mathbf{R}$ be a continuous function. Suppose $f(a) < 0 < f(b)$. Then there is some point $c \in (a,b)$ with $f(c) = 0$.*

Proof: We will construct a binary search sequence $[a_n, b_n]$ with $[a_0, b_0] = [a, b]$ such that

$$f(a_n) \le 0 \le f(b_n) \text{ for all } n. \tag{9.12}$$

Let

$$\begin{aligned} [a_0, b_0] &= [a, b] \\ [a_{n+1}, b_{n+1}] &= \begin{cases} \left[a_n, \frac{a_n+b_n}{2}\right] & \text{if } f\left(\frac{a_n+b_n}{2}\right) \ge 0 \\ \left[\frac{a_n+b_n}{2}, b_n\right] & \text{if } f\left(\frac{a_n+b_n}{2}\right) < 0. \end{cases} \end{aligned}$$

This is a binary search sequence satisfying condition (9.12).

Let $c$ be the number such that $\{[a_n, b_n]\} \to c$. Then $\{a_n\} \to c$ and $\{b_n\} \to c$ (cf theorem 7.87), so by continuity of $f$, $\{f(a_n)\} \to f(c)$ and $\{f(b_n)\} \to f(c)$. Since $f(b_n) \ge 0$ for all $n$, it follows by the inequality theorem that $f(c) = \lim\{f(b_n)\} \ge 0$, and since $f(a_n) \le 0$, we have $f(c) = \lim\{f(a_n)\} \le 0$. Hence, $f(c) = 0$. |||

**9.13 Exercise (Intermediate value theorem.)** Let $a, b \in \mathbf{R}$ with $a < b$ and let $f : [a,b] \to \mathbf{R}$ be a continuous function with $f(a) < f(b)$. Let $y$ be a number in the interval $(f(a), f(b))$. Show that there is some $c \in (a,b)$ with $f(c) = y$. (Use theorem 9.11. Do not reprove it.)

**9.14 Notation ($x$ is between $a$ and $b$.)** Let $a, b, x \in \mathbf{R}$. I say $x$ is *between* $a$ and $b$ if either $a < x < b$ or $b < x < a$.

**9.15 Corollary (Intermediate value theorem.)** *Let $a, b \in \mathbf{R}$ with $a < b$. Let $f\colon [a,b] \to \mathbf{R}$ be a continuous function with $f(a) \neq f(b)$. If $y$ is any number between $f(a)$ and $f(b)$, then there is some $c \in (a,b)$ such that $f(c) = y$. In particular, if $f(a)$ and $f(b)$ have opposite signs, there is a number $c \in (a,b)$ with $f(c) = 0$.*

Proof: By exercise 9.13, the result holds when $f(a) < f(b)$. If $f(a) > f(b)$, let $g = -f$. Then $g$ is continuous on $[a,b]$ and $g(a) < g(b)$, so by exercise 9.13 there is a $c \in (a,b)$ with $g(c) = 0$, so $-f(c) = 0$ so $f(c) = 0$. $\|$

**9.16 Example.** Let $A, B, C, D$ be real numbers with $A \neq 0$, and let

$$f(x) = Ax^3 + Bx^2 + Cx + D.$$

We will show that there is a number $c \in \mathbf{R}$ such that $f(c) = 0$. Suppose, in order to get a contradiction, that no such number $c$ exists, and let

$$g(x) = \frac{f(-x)}{f(x)} = \frac{-Ax^3 + Bx^2 - Cx + D}{Ax^3 + Bx^2 + C + D} \text{ for all } x \in \mathbf{R}.$$

(I use the fact that $f(x)$ has no zeros here.) Then

$$\begin{aligned} \lim\{g(n)\}_{n\geq 1} &= \lim\left\{\frac{-A + \frac{B}{n} - \frac{C}{n^2} + \frac{D}{n^3}}{A + \frac{B}{n} + \frac{C}{n^2} + \frac{D}{n^3}}\right\}_{n\geq 1} \\ &= \frac{-A + 0 + 0 + 0}{A + 0 + 0 + 0} = -1. \end{aligned}$$

It follows that $g(n) < 0$ for some $n$, so $f(-n)$ and $f(n)$ have opposite signs for some $n$, and $g$ is continuous on $[-n, n]$, so by the intermediate value theorem, $g(c) = 0$ for some $c \in (-n, n)$, contradicting the assumption that $g$ is never zero.

**9.17 Exercise.** Give examples of the requested functions, or explain why no such function exists. Describe your functions by formulas if you can, but pictures of graphs will do if a formula seems too complicated.

a) $f\colon [0,1] \to \mathbf{R}$, $f$ has no maximum.

b) $g\colon [0,\infty) \to \mathbf{R}$, $g$ is continuous, $g$ has no maximum.

c) $k\colon [0,\infty) \to \mathbf{R}$, $k$ is continuous, $k$ has no maximum or minimum.

d) $l: (0,1) \to \mathbf{R}$, $l$ is bounded and continuous, $l$ has no maximum.

**9.18 Exercise.** Let $f(x) = x^3 - 3x + 1$. Prove that the equation $f(x) = 0$ has at least three solutions in $\mathbf{R}$.

**9.19 Exercise.** Let $F$ be a continuous function from $\mathbf{R}$ to $\mathbf{R}$ such that

a) For all $x \in \mathbf{R}$, $\Big((F(x) = 0) \iff (x^2 = 1)\Big)$.

b) $F(2) > 0$.

Prove that $F(4) > 0$.

**9.20 Note.** The intermediate value theorem was proved independently by Bernhard Bolzano in 1817 [42], and Augustin Cauchy in 1821[23, pp 167-168]. The proof we have given is almost identical with Cauchy's proof.

The extreme value theorem was proved by Karl Weierstrass circa 1861.