Chapter 7

Complex Sequences

In definition 5.1, we defined a sequence in $\mathbf{C}$ to be a function $f: \mathbf{N} \to \mathbf{C}$. Since we are identifying $\mathbf{R}$ with a subset of $\mathbf{C}$, every sequence in $\mathbf{R}$ is also a sequence in $\mathbf{C}$, and all of our results for complex sequences are applicable to real sequences.

7.1 Some Examples.

7.1 Notation ($\mapsto$) I will say "consider the sequence $n \mapsto 2^n$" or "consider the sequence $f: n \mapsto 2^n$" to mean "consider the sequence $f: \mathbf{N} \to \mathbf{C}$ such that $f(n) = 2^n$ for all $n \in \mathbf{N}$". The arrow $\mapsto$ is read "maps to".

7.2 Definition (Geometric sequence.) For each $\alpha \in \mathbf{C}$, the sequence

$$n \mapsto \alpha^n$$

is called the *geometric sequence with ratio* α.

I will often represent a sequence f in $\mathbf{C}$ by a polygonal line with vertices $f(0), f(1), f(2), \cdots$. The two figures below represent geometric sequences with ratios $\frac{1+i}{2}$ and $\frac{2+i}{3}$ respectively.

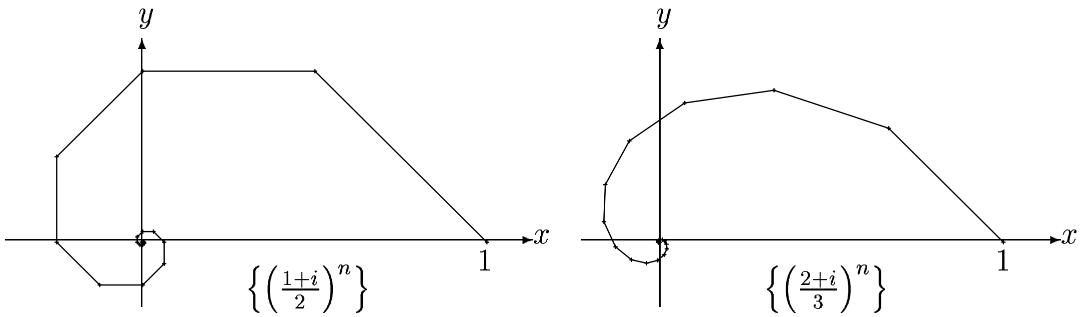
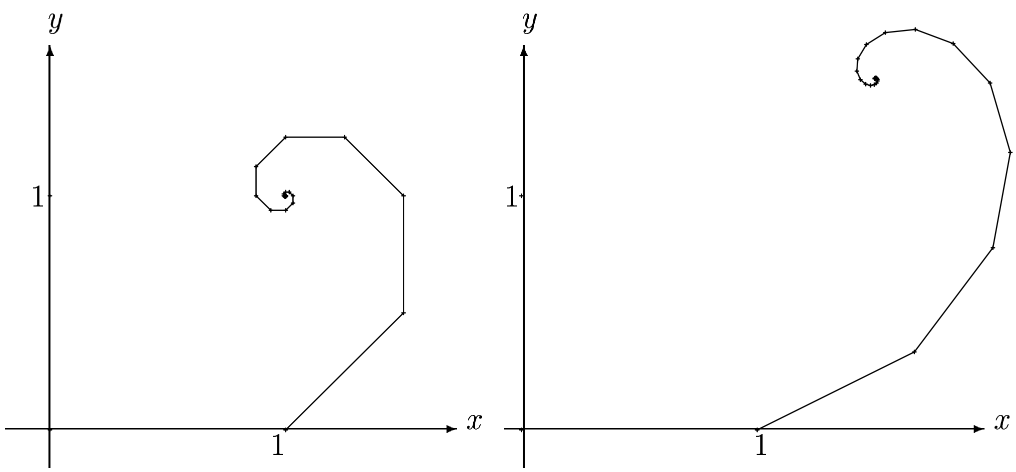


Figure a. Geometric Sequences

7.3 Definition (Geometric series.) If $\alpha \in \mathbf{C}$, then the sequence $g_\alpha : n \mapsto \sum_{j=0}^{n} \alpha^j$ is called the *geometric series with ratio* α.

$$g_\alpha = \{1, 1+\alpha, 1+\alpha+\alpha^2, 1+\alpha+\alpha^2+\alpha^3, \cdots\}$$



$\alpha = \frac{1+i}{2}$ $\alpha = \frac{2+i}{3}$

Figure b. Geometric Series $\left\{\sum_{j=0}^{n} \alpha^j\right\}$

Figure b shows the geometric series corresponding to the geometric sequences in figure a. If you examine the figures you should notice a remarkable similarity between the figure representing $\{\alpha^n\}$ and the figure representing $\{\sum_{j=0}^{n} \alpha^j\}$.

7.4 Entertainment. Describe the apparent similarity between the figure for $\{\alpha^n\}$ and the figure for $\{\sum_{j=0}^{n} \alpha^j\}$. Then prove that this similarity is really present for all $\alpha \in \mathbf{C}\backslash\{1\}$.

7.5 Definition (Constant sequence.) For each $\alpha \in \mathbf{C}$, let $\tilde{\alpha}$ denote the constant sequence $\tilde{\alpha}: n \mapsto \alpha$; i.e., $\tilde{\alpha} = \{\alpha, \alpha, \alpha, \alpha, \cdots\}$.

7.2 Convergence

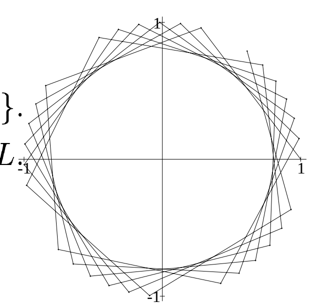
7.6 Definition (Convergent sequence.) Let f be a complex sequence, and let $L \in \mathbf{C}$. We will say f *converges to* L and write $f \to L$ if for every disc $D(L, r)$ there is a number $N \in \mathbf{N}$ such that

$$\text{for every } n \in \mathbf{Z}_{\geq N}, \quad (f(n) \in D(L, r)).$$

We say f *converges* if there is some $L \in \mathbf{C}$ such that $f \to L$. We say f *diverges* if and only if f does not converge.

It appears from figure a on page 126 that for every disc $D(0, r)$ centered at 0 the terms of the sequence $\left\{\left(\frac{1+i}{2}\right)^n\right\}$ eventually get into $D(0, r)$; i.e., it appears that $\left\{\left(\frac{1+i}{2}\right)^n\right\} \to 0$. Similarly, it appears that $\left\{\left(\frac{1+2i}{3}\right)^n\right\} \to 0$.

From figure b, it appears that there are numbers P, Q such that $\left\{\sum_{j=0}^{n}\left(\frac{1+i}{2}\right)^j\right\} \to P$, and $\left\{\sum_{j=0}^{n}\left(\frac{1+2i}{3}\right)^j\right\} \to Q$. You should be able to put your finger on P and Q, and maybe to guess what their exact values are. We will return to these examples later.



$\left\{\left(\frac{7+24i}{25}\right)^n\right\}$

Let $w = \frac{7+24i}{25}$. The figure in the margin represents the sequence $\{w^n\}$. It appears from the figure that there is no number L such that $\{w^n\} \mapsto L$. The following theorem shows that this is the case.
(Note that $\left|\frac{7+24i}{25}\right| = \sqrt{\frac{49+576}{625}} = 1$.)

7.7 Theorem. *Let $w \in \mathbf{C}$ satisfy $|w| \geq 1$ and $w \neq 1$. Then $\{w^n\}$ diverges.*

Proof: Suppose that $|w| \geq 1$ and $w \neq 1$. Then for all $n \in \mathbf{N}$,

$$|w^n - w^{n+1}| = |w^n(1-w)| = |w|^n|1-w| \geq |1-w| > 0. \tag{7.8}$$

Now suppose, to get a contradiction, that there is a number $L \in \mathbf{C}$ such that $\{w^n\} \to L$. Then corresponding to the disc $D\left(L, \dfrac{|1-w|}{2}\right)$, there is a number $N \in \mathbf{N}$ such that

$$n \in \mathbf{Z}_{\geq N} \implies w^n \in D\left(L, \frac{|1-w|}{2}\right).$$

In particular,

$$w^N \in D\left(L, \frac{|1-w|}{2}\right) \text{ and } w^{N+1} \in D\left(L, \frac{|1-w|}{2}\right)$$

so

$$|w^N - L| < \frac{|1-w|}{2} \text{ and } |w^{N+1} - L| < \frac{|1-w|}{2}.$$

By the triangle inequality,

$$\begin{aligned} |w^N - w^{N+1}| &= |(w^N - L) + (L - w^{N+1})| \\ &\leq |w^N - L| + |L - w^{N+1}| \\ &< \frac{|1-w|}{2} + \frac{|1-w|}{2} = |1-w|. \end{aligned}$$

Combining this result with (7.8), we get

$$|1-w| \leq |w^N - w^{N+1}| < |1-w|,$$

so $|1-w| < |1-w|$. This contradiction shows that $\{w^n\}$ diverges. ‖

We can also show that constant sequences converge.

7.9 Theorem. *Let $\alpha \in \mathbf{C}$. Then the constant sequence $\tilde{\alpha}$ converges to α.*

Proof: Let $\alpha \in \mathbf{C}$. Let $D(\alpha, r)$ be a disc centered at α. Then

$$\tilde{\alpha}(n) = \alpha \in D(\alpha, r) \text{ for all } n \in \mathbf{Z}_{\geq 0},$$

Hence, $\tilde{\alpha} \to \alpha$. ‖

For purposes of calculation it is sometimes useful to rephrase the definition of convergence. Since the disc $D(\alpha, r)$ is determined by its radius r, and for all $z \in \mathbf{C}$, $z \in D(\alpha, r) \iff |z - \alpha| < r$, we can reformulate definition 7.6 as

7.10 Definition (Convergence.) Let f be a sequence in $\mathbf{C}$, and let $L \in \mathbf{C}$. Then $f \to L$ if and only if for every $r \in \mathbf{R}^+$ there is some $N \in \mathbf{N}$ such that

$$\text{for every } n \in \mathbf{Z}_{\geq N}, \quad (|f(n) - L| < r).$$

7.3 Null Sequences

Sequences that converge to 0 are simpler to work with than general sequences, and many of the convergence theorems for general sequences can be easily deduced from the properties of sequences that converge to 0. In this section we will just consider sequences that converge to 0.

7.11 Definition (Null sequence.) Let f be a sequence in $\mathbf{C}$. We will say f is a *null sequence* if and only if for every $\varepsilon \in \mathbf{R}^+$ there is some $N \in \mathbf{N}$ such that for every $n \in \mathbf{Z}_{\geq N}$, $(|f(n)| < \varepsilon)$.

By comparing this definition with definition 7.10, you see that

$$(f \text{ is a null sequence }) \iff (f \to 0).$$

Definition 7.11 is important. You should memorize it.

7.12 Definition (Dull sequence.) Let f be a sequence in $\mathbf{C}$. We say f is a *dull sequence* if and only if there is some $N \in \mathbf{N}$ such that for every ε in $\mathbf{R}^+$, and for every $n \in \mathbf{Z}_{\geq N}$ $(|f(n)| < \varepsilon)$.

The definitions of null sequence and dull sequence use the same words, but they are not in the same order, and the definitions are not equivalent.

If f satisfies condition (7.12), then whenever $n \geq N$,

$$\text{for every } \varepsilon \text{ in } \mathbf{R}^+ \quad (|f(n)| < \varepsilon)\,.$$

If $|f(n)| \in \mathbf{R}^+$, this condition would say $|f(n)| < |f(n)|$, which is false. Hence if $n \geq N$, then $|f(n)| \notin \mathbf{R}^+$; i.e., if $n \geq N$, then $f(n) = 0$. Hence a dull sequence has the property that there is some $N \in \mathbf{N}$ such that $f(n) = 0$ for all $n \geq N$. Thus every dull sequence is a null sequence. The sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, 0, 0, 0, 0, \cdots\right\}$$

is a dull sequence, but

$$\left\{\frac{1}{n}\right\}_{n\geq 1} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \cdots\right\}$$

is not a dull sequence. In the next theorem we show that $\{\frac{1}{n}\}_{n\geq 1}$ is a null sequence, so null sequences are not necessarily dull.

7.13 Theorem. *For all $a \in \mathbf{C}$, $\left\{\frac{a}{n}\right\}_{n\geq 1}$ is a null sequence .*

Proof: Let $\varepsilon \in \mathbf{R}^+$. By the Archimedean property for $\mathbf{R}$, there is an $N \in \mathbf{Z}^+$ such that $N > \frac{|a|}{\varepsilon}$. Then for all $n \in \mathbf{Z}^+$,

$$n \geq N \implies n > \frac{|a|}{\varepsilon} \implies \frac{|a|}{n} < \varepsilon,$$

so for all $n \in \mathbf{Z}_{\geq N}$ $\left(\left|\frac{a}{n}\right| < \varepsilon\right)$. ‖

The difference between a null sequence and a dull sequence is that the "N" in the definition of null sequence can (and usually does) depend on ε, while the "N" in the definition of dull sequence depends only on f. To emphasize that N depends on ε (and also on f), I will often write $N(\varepsilon)$ or $N_f(\varepsilon)$ instead of N.

Here is another reformulation of the definition of null sequence.

7.14 Definition (Precision function.) Let f be a complex sequence. Then f is a null sequence if and only if there is a function $N_f \colon \mathbf{R}^+ \to \mathbf{N}$ such that

$$\text{for all } \varepsilon > 0 \text{ and all } n \in \mathbf{N};\ (n \geq N_f(\varepsilon) \implies |f(n)| < \varepsilon).$$

I will call such a function N_f a *precision function for f*.

This formulation shows that in order to show that a sequence f is a null sequence, you need to find a *function* $N_f \colon \mathbf{R}^+ \to \mathbf{N}$ such that

$$\text{for all } n \in \mathbf{N}\ \ (n \geq N_f(\varepsilon) \implies |f(n)| < \varepsilon).$$

In the proof of theorem 7.13, for the sequence $g\colon n \mapsto \dfrac{a}{n}$ we had

$$N_g(\varepsilon) = \left(\text{some integer } N \text{ such that } \frac{|a|}{N} < \varepsilon\right).$$

This description for N_g could be made more precise, but it is good enough for our purposes.

7.15 Theorem. *If $\alpha \in \mathbf{C}\backslash\{0\}$, then the constant sequence $\tilde{\alpha}$ is not a null sequence.*

Proof: If $\alpha \neq 0$, then $\frac{1}{2}|\alpha| \in \mathbf{R}^+$. Suppose, to get a contradiction, that $\tilde{\alpha}$ is a null sequence. Then there is a number $N \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ $\left(n \geq N \implies |\tilde{\alpha}(n)| < \frac{1}{2}|\alpha|\right)$. Then for all $n \in \mathbf{N}$,

$$\left(n \geq N \implies |\alpha| < \frac{1}{2}|\alpha| \implies 1 < \frac{1}{2}\right). \tag{7.16}$$

If $n = N+1$ then (7.16) is false and this shows that $\tilde{\alpha}$ is not a null sequence. |||

7.17 Theorem (Comparison theorem for null sequences.) *Let f, g be complex sequences. Suppose that f is a null sequence and that*

$$|g(n)| \leq |f(n)| \text{ for all } n \in \mathbf{N}.$$

Then g is a null sequence.

Proof: Since f is a null sequence, there is a function $N_f\colon \mathbf{R}^+ \to \mathbf{N}$ such that for all $n \in \mathbf{N}$,

$$n \geq N_f(\varepsilon) \implies |f(n)| < \varepsilon.$$

Then

$$n \geq N_f(\varepsilon) \implies |g(n)| \leq |f(n)| < \varepsilon \implies |g(n)| < \varepsilon.$$

Hence, we can let $N_g = N_f$. |||

7.18 Example. We know that $n \leq 2^n$ for all $n \in \mathbf{N}$, so $\dfrac{1}{2^n} \leq \dfrac{1}{n}$ for all $n \in \mathbf{Z}_{\geq 1}$. Since $\left\{\dfrac{1}{n}\right\}_{n\geq 1}$ is a null sequence, it follows from the comparison theorem that $\left\{\dfrac{1}{2^n}\right\}_{n\geq 1}$ is a null sequence. Also, since $\dfrac{1}{n^2+n} \leq \dfrac{1}{n}$ for all $n \in \mathbf{Z}_{\geq 1}$, we see that $\left\{\dfrac{1}{n^2+n}\right\}_{n\geq 1}$ is a null sequence.

7.19 Theorem (Root theorem for null sequences.)

Let $f\colon \mathbf{N} \to [0,\infty)$ be a null sequence, and let $p \in \mathbf{Z}_{\geq 1}$. Then $f^{\frac{1}{p}}$ is a null sequence where $f^{\frac{1}{p}}(n) = (f(n))^{\frac{1}{p}}$ for all $n \in \mathbf{N}$.

Scratchwork: Let $g = f^{\frac{1}{p}}$. I want to find N_g so that for all $n \in \mathbf{N}$ and all $\varepsilon \in \mathbf{R}^+$,

$$n \geq N_g(\varepsilon) \implies |g(n)| \leq \varepsilon$$

i.e.

$$n \geq N_g(\varepsilon) \implies \left|f^{\frac{1}{p}}(n)\right| \leq \varepsilon$$

i.e.

$$n \geq N_g(\varepsilon) \implies f(n) \leq \varepsilon^p.$$

This suggests that I should take $N_g(\varepsilon) = N_f(\varepsilon^p)$.

Proof: Let f be a null sequence in $[0,\infty)$ and let N_f be a precision function for f. Define $N_g\colon \mathbf{R}^+ \to \mathbf{N}$ by $N_g(\varepsilon) = N_f(\varepsilon^p)$ for all $\varepsilon \in \mathbf{R}^+$. Then for all $n \in \mathbf{N}$,

$$\begin{aligned} n \geq N_g(\varepsilon) &\implies n \geq N_f(\varepsilon^p) \\ &\implies |f(n)| < \varepsilon^p \\ &\implies 0 \leq f(n) < \varepsilon^p \\ &\implies f(n)^{1/p} < \varepsilon \\ &\implies g(n) < \varepsilon. \end{aligned}$$

Hence N_g is a precision function for g. |||

7.20 Examples. Let $c \in \mathbf{R}^+$. Then $\left\{\dfrac{c^2}{n}\right\}_{n\geq 1}$ is a null sequence in $[0,\infty)$, so it follows that $\left\{\dfrac{c}{\sqrt{n}}\right\}_{n\geq 1}$ is a null sequence.

Consider the sequence $f\colon \mathbf{Z}_{\geq 1} \to \mathbf{C}$, $f : n \mapsto n + \dfrac{1}{2} - \sqrt{n^2+n}$. For all $n \in \mathbf{Z}_{\geq 1}$,

$$\begin{aligned} f(n) &= \left(\left(n + \frac{1}{2}\right) - \sqrt{n^2+n}\right) \frac{\left(\left(n+\frac{1}{2}\right) + \sqrt{n^2+n}\right)}{\left(\left(n+\frac{1}{2}\right) + \sqrt{n^2+n}\right)} \\ &= \frac{\left(n^2+n+\frac{1}{4}\right) - (n^2+n)}{n+\frac{1}{2}+\sqrt{n^2+n}} = \frac{1}{4\left(n+\frac{1}{2}+\sqrt{n^2+n}\right)}. \end{aligned}$$

Hence $|f(n)| \leq \dfrac{1}{4n} \leq \dfrac{1}{n}$, so it follows from the comparison theorem that f is a null sequence.

Since $\left\{\dfrac{1}{2^n}\right\}_{n\geq 1}$ is a null sequence, it follows from the root theorem that $\left\{\left(\dfrac{1}{\sqrt{2}}\right)^n\right\}_{n\geq 1}$ is a null sequence. Now $.7^2 = .49 < \dfrac{1}{2}$, so $.7 < \sqrt{\dfrac{1}{2}} = \dfrac{1}{\sqrt{2}}$ so $(.7)^n \leq \left(\dfrac{1}{\sqrt{2}}\right)^n$ for all $n \in \mathbf{Z}_{\geq 1}$, and by another comparison test, $\{.7^n\}$ is a null sequence. Since $\big((|\alpha| \leq .7) \implies (|\alpha^n| \leq .7^n)\big)$, it follows that $\{\alpha^n\}_{n\geq 1}$ is a null sequence for all $\alpha \in \mathbf{C}$ with $|\alpha| \leq .7$.

You probably suspect that $\{\alpha^n\}$ is a null sequence for all $\alpha \in \mathbf{C}$ with $|\alpha| < 1$. This is correct, but we will not prove it yet.

7.21 Exercise. Show that the geometric sequences $\left\{\left(\dfrac{1+i}{2}\right)^n\right\}$ and $\left\{\left(\dfrac{2+i}{3}\right)^n\right\}$ that are sketched on page 126 are in fact null sequences.

7.22 Exercise. Which, if any, of the sequences below are null sequences? Justify your answers.

a) $\{\sqrt{n+10000} - \sqrt{n}\}_{n\geq 1}$

b) $\left\{\dfrac{n^2+1}{n^3+3n}\right\}_{n\geq 1}$

c) $\left\{\dfrac{n^2+6}{n^3+3n}\right\}_{n\geq 1}$

7.23 Entertainment. Show that

$$\{(1-10^{-20})^n\} = \{.99999999999999999999^n\}$$

is a null sequence. (If you succeed, you will probably find a proof that $\{\alpha^n\}$ is a null sequence whenever $|\alpha| < 1$.) NOTE: If you use calculator operations, then $\{(1-10^{-20})^n\}$ is not a null sequence on most calculators.

It follows from remark 5.38 that we can add, subtract and multiply complex sequences, and that the usual associative, commutative, and distributive laws hold. If $f = \{f(n)\}$ and $g = \{g(n)\}$ then $f + g = \{f(n) + g(n)\}$ and $(fg)(n) = \{f(n) \cdot g(n)\}$. If $\alpha, \beta \in \mathbf{C}$ then the constant sequences $\tilde{\alpha}, \tilde{\beta}$ satisfy

$$\widetilde{\alpha + \beta} = \tilde{\alpha} + \tilde{\beta}, \quad \widetilde{\alpha\beta} = \tilde{\alpha}\tilde{\beta}.$$

7.24 Exercise. Which of the field axioms are satisfied by addition and multiplication of sequences? Does the set of complex sequences form a field? (You know that the associative, distributive and commutative laws hold, so you just need to consider the remaining axioms.

7.25 Notation. If f is a complex sequence, we define sequences f^*, $\text{Re}f$, $\text{Im}f$, and $|f|$ by

$$\begin{aligned} f^*(n) &= (f(n))^* \text{ for all } n \in \mathbf{N}, \\ (\text{Re}f)(n) &= \text{Re}\,(f(n)) \text{ for all } n \in \mathbf{N}, \\ (\text{Im}f)(n) &= \text{Im}\,(f(n)) \text{ for all } n \in \mathbf{N}, \\ |f|(n) &= |f(n)| \text{ for all } n \in \mathbf{N}. \end{aligned}$$

7.26 Theorem. *Let f be a complex null sequence. Then f^*, $\text{Re}f$, $\text{Im}f$ and $|f|$ are all null sequences.*

Proof: All four results follow by the comparison theorem. We have, for all $n \in \mathbf{N}$:

$$\begin{aligned} |f^*(n)| &= |(f(n))^*| = |f(n)|, \\ |(\text{Re}f)(n)| &= |\text{Re}(f(n))| \leq |f(n)|, \\ |(\text{Im}f)(n)| &= |\text{Im}(f(n))| \leq |f(n)|, \\ ||f|(n)| &= |f(n)|. \ ||| \end{aligned}$$

7.4 Sums and Products of Null Sequences

7.27 Theorem (Sum theorem for null sequences.) *Let f, g be complex null sequences and let $\alpha \in \mathbf{C}$. Then $f + g$, $f - g$, and αf are null sequences.*

Scratchwork for αf: I want to find $N_{\alpha f}$ so that

$$n \geq N_{\alpha f}(\varepsilon) \implies |\alpha f(n)| < \varepsilon$$

i.e.

$$n \geq N_{\alpha f}(\varepsilon) \implies |f(n)| < \frac{\varepsilon}{|\alpha|}.$$

This suggests that I take $N_{\alpha f}(\varepsilon) = N_f\left(\frac{\varepsilon}{|\alpha|}\right)$.

Scratchwork for $f+g$: I want to find N_{f+g} so that

$$n \geq N_{f+g}(\varepsilon) \implies |f(n)+g(n)| < \varepsilon.$$

Now $|f(n)+g(n)| \leq |f(n)|+|g(n)|$, and I can make $|f(n)|+|g(n)| < \varepsilon$ by making $|f(n)| < \varepsilon/2$ and $|g(n)| < \varepsilon/2$. Hence I want $N_{f+g}(\varepsilon) > N_f(\varepsilon/2)$ and $N_{f+g}(\varepsilon) > N_g\left(\frac{\varepsilon}{2}\right)$. This suggests that I take $N_{f+g}(\varepsilon) = \max\left(N_f(\varepsilon/2), N_g(\varepsilon/2)\right)$.

Proof: Let f, g be null sequences, and let $\alpha \in \mathbf{C}$. Define $N_{f+g}\colon \mathbf{R}^+ \to \mathbf{N}$ by

$$N_{f+g}(\varepsilon) = \max\left(N_f(\varepsilon/2), N_g(\varepsilon/2)\right).$$

Then for all $n \in \mathbf{N}$,

$$\begin{aligned} n \geq N_{f+g}(\varepsilon) &\implies n \geq N_f(\varepsilon/2) \text{ and } n \geq N_g(\varepsilon/2) \\ &\implies |f(n)| < \varepsilon/2 \text{ and } |g(n)| < \varepsilon/2 \\ &\implies |f(n)+g(n)| \leq |f(n)|+|g(n)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &\implies |(f+g)(n)| < \varepsilon. \end{aligned}$$

Hence, N_{f+g} is a precision function for $f+g$, and $f+g$ is a null sequence.

If $\alpha = 0$ then $\alpha f = \tilde{0}$ is a null sequence. Suppose $\alpha \neq 0$, and define $N_{\alpha f}\colon \mathbf{R}^+ \to \mathbf{N}$ by

$$N_{\alpha f}(\varepsilon) = N_f\left(\frac{\varepsilon}{|\alpha|}\right).$$

Then for all $n \in \mathbf{Z}$,

$$\begin{aligned} n \geq N_{\alpha f} &\implies n \geq N_f\left(\frac{\varepsilon}{|\alpha|}\right) \\ &\implies |f(n)| \leq \frac{\varepsilon}{|\alpha|} \\ &\implies |\alpha|\,|f(n)| \leq \varepsilon \\ &\implies |\alpha f(n)| \leq \varepsilon. \end{aligned}$$

Hence $N_{\alpha f}$ is a precision function for αf, and hence αf is a null sequence. Since $f - g = f + (-1)g$ it follows that $f - g$ is a null sequence. |||

7.28 Exercise (Product theorem for null sequences.) Let f, g be complex null sequences. Prove that fg is a null sequence.

7.5 Theorems About Convergent Sequences

7.29 Remark. *Let f be a complex sequence, and let $L \in \mathbf{C}$. Then the following three statements are equivalent.*

a) $f \to L$

b) $f - \tilde{L}$ *is a null sequence.*

c) $|f - \tilde{L}|$ *is a null sequence.*

Proof: By definition 7.10, "$f \to L$" means

$$\text{for every } r \in \mathbf{R}^+ \text{ there is some } N \in \mathbf{N} \text{ such that}$$
$$\text{for every } n \in \mathbf{Z}_{\geq N},\ (|f(n) - L| < r).$$

By definition 7.11, "$f - \tilde{L}$ is a null sequence" means

$$\text{for every } \varepsilon \in \mathbf{R}^+ \text{ there is some } N \in \mathbf{N} \text{ such that}$$
$$\text{for every } \ n \in \mathbf{Z}_{\geq N}, |(f - \tilde{L})(n)| < \varepsilon. \tag{7.30}$$

Both definitions say the same thing. If we write out the definition for "$|f - \tilde{L}|$ is a null sequence" we get (7.30) with "$|(f - \tilde{L})(n)| < \varepsilon$" replaced by "$\Big||f - \tilde{L}|(n)\Big| < \varepsilon$." Since

$$|(f - \tilde{L})(n)| = |f(n) - L| = \Big||f - \tilde{L}|(n)\Big|,$$

conditions b) and c) are equivalent. |||

7.31 Theorem (Decomposition theorem.) *Let f be a convergent complex sequence. Then we can write*

$$f = k + \tilde{c}$$

where k is a null sequence, and $\tilde{c}$ is a constant sequence. If $f \to L$, then $c = L$.

Proof: $f = (f - \tilde{L}) + \tilde{L}$. ‖

7.32 Theorem (Sum theorems for convergent sequences.) *Let $\alpha \in \mathbf{C}$ and let f, g be convergent complex sequences. Say $f \to L$ and $g \to M$. Then $f+g$, $f-g$ and αf are convergent and*

$$\begin{aligned} f+g &\to L+M \\ f-g &\to L-M \\ \alpha f &\to \alpha L. \end{aligned}$$

Proof: Suppose $f \to L$ and $g \to M$. By the decomposition theorem, we can write

$$f = k + \tilde{L} \text{ and } g = p + \tilde{M}$$

where k and p are null sequences. Then

$$(f \pm g) - (\widetilde{L \pm M}) = (k + \tilde{L}) \pm (p + \tilde{M}) - (\tilde{L} \pm \tilde{M}) = k \pm p.$$

By the sum theorem for null sequences, $k \pm p$ is a null sequence, so $(f \pm g) - \widetilde{L \pm M}$ is a null sequence, and hence $f \pm g \to L \pm M$. ‖

7.33 Exercise. Prove the last statement in theorem 7.32; i.e., show that if $f \to L$ then $\alpha f \to \alpha L$ for all $\alpha \in \mathbf{C}$.

7.34 Theorem (Product theorem for convergent sequences.) *Let f, g be convergent complex sequences. Suppose $f \to L$ and $g \to M$. Then fg is convergent and $fg \to LM$.*

Proof: Suppose $f \to L$ and $g \to M$. Write $f = k + \tilde{L}$, $g = p + \tilde{M}$ where k, p are null sequences. Then

$$\begin{aligned} fg &= (k + \tilde{L})(p + \tilde{M}) \\ &= kp + \tilde{L}p + \tilde{M}k + \tilde{L}\tilde{M} \\ &= kp + Lp + Mk + \widetilde{LM}. \end{aligned}$$

Now kp, Lp and Mk are null sequences by the product theorem and sum theorem for null sequences, and $\widetilde{LM} \to LM$, so by several applications of the sum theorem for convergent sequences,

$$fg \to 0 + 0 + 0 + LM; \text{ i.e. } fg \to LM. \ \|$$

7.35 Theorem (Uniqueness theorem for convergent sequences.) *Let f be a complex sequence, and let $L, M \in \mathbf{C}$. If $f \to L$ and $f \to M$, then $L = M$.*

Proof: Suppose $f \to L$ and $f \to M$. Then $f - \tilde{L}$ and $f - \tilde{M}$ are null sequences, so $(f - \tilde{L}) - (f - \tilde{M}) = \tilde{M} - \tilde{L} = \widetilde{M - L}$ is a null sequence. Hence, by theorem 7.15, $M - L = 0$; i.e., $L = M$. |||

7.36 Definition (Limit of a sequence.) Let f be a convergent sequence. Then the unique complex number L such that $f \to L$ is denoted by $\lim f$ or $\lim\{f(n)\}$.

7.37 Remark. It follows from the sum and product theorems that if f and g are convergent sequences, then

$$\lim(f \pm g) = \lim f \pm \lim g$$

and

$$\lim(f \cdot g) = \lim f \cdot \lim g$$

and

$$\lim cf = c \lim f.$$

7.38 Warning. We have only defined $\lim f$ when f is a convergent sequence. Hence $\lim\{i^n\}$ is ungrammatical and should not be written down. (We showed in theorem 7.7 that $\{i^n\}$ diverges.) However, it is a standard usage to say "$\lim f$ does not exist" or "$\lim\{f(n)\}$ does not exist" to mean that the sequence f has no limit. Hence we may say "$\lim\{i^n\}$ does not exist".

7.39 Theorem. *Let f be a complex sequence. Then f is convergent if and only if both* $\text{Re}f$ *and* $\text{Im}f$ *are convergent. Moreover,*

$$\begin{aligned} \lim f &= \lim \text{Re}f + i \lim \text{Im}f, \qquad (7.40) \\ \lim \text{Re}f &= \text{Re}(\lim f), \\ \lim \text{Im}f &= \text{Im}(\lim f). \end{aligned}$$

Proof: If Ref and Imf are convergent, then it follows from the sum theorem for convergent sequences that f is convergent and (7.40) is valid.

Suppose that $f \to L$. Then $f - \tilde{L}$ is a null sequence, so $\text{Re}(f - \tilde{L})$ is a null sequence (by Theorem 7.26). For all $n \in \mathbf{N}$,

$$\text{Re}(f - \tilde{L})(n) = \text{Re}\,(f(n) - L) = \text{Re}f(n) - \text{Re}L = (\text{Re}f - \widetilde{\text{Re}L})(n)$$

so $(\text{Re}f - \widetilde{\text{Re}L}) = \text{Re}(f - \tilde{L})$ is a null sequence and it follows that Ref converges to ReL. A similar argument shows that $\text{Im}f \to \text{Im}L$. |||

7.41 Definition (Bounded sequence.) A sequence f in $\mathbf{C}$ is *bounded*, if there is a disc $\overline{D}(0, B)$ such that $f(n) \in \overline{D}(0, B)$ for all $n \in \mathbf{N}$; i.e., f is bounded if there is a number $B \in [0, \infty)$ such that

$$|f(n)| \leq B \text{ for all } n \in \mathbf{N}. \tag{7.42}$$

Any number B satisfying condition (7.42) is called a *bound for f*.

7.43 Examples. $\left\{\dfrac{i^n n}{n+1}\right\}$ is bounded since $\left|\dfrac{i^n n}{n+1}\right| = \dfrac{n}{n+1} \leq 1$ for all $n \in \mathbf{N}$. The sequence $\{n\}$ is not bounded since the statement $|n| \leq B$ for all $n \in \mathbf{N}$ contradicts the Archimedean property of $\mathbf{R}$. Every constant sequence $\{\tilde{L}\}$ is bounded. In fact, $|L|$ is a bound for $\tilde{L}$.

7.44 Exercise (Null-times-bounded theorem.) Show that if f is a null sequence in $\mathbf{C}$, and g is a bounded sequence in $\mathbf{C}$ then fg is a null sequence.

The next theorem I want to prove is a quotient theorem for convergent sequences. To prove this, I will need some technical results.

7.45 Theorem (Reverse triangle inequality.) *Let $\alpha, \beta \in \mathbf{C}$, then*

$$|\alpha - \beta| \geq |\alpha| - |\beta|.$$

Proof: By the triangle inequality.

$$|\alpha| = |(\alpha - \beta) + \beta| \leq |\alpha - \beta| + |\beta|.$$

Hence,

$$|\alpha| - |\beta| \leq |\alpha - \beta|. \ |||$$

7.46 Lemma. *Let f be a convergent sequence that is not a null sequence; i.e., $f \to L$ where $L \neq 0$. Suppose $f(n) \neq 0$ for all $n \in \mathbf{N}$. Then $\frac{1}{f}$ is a bounded sequence.*

Proof: Since $f \to L$, we know that $f - \tilde{L}$ is a null sequence. Let $N_{f-\tilde{L}}$ be a precision function for $f - \tilde{L}$. Then for all $n \in \mathbf{N}$,

$$\begin{aligned} n \geq N_{f-\tilde{L}}\left(\frac{|L|}{2}\right) &\implies |f(n) - L| < \frac{|L|}{2} \\ &\implies \frac{|L|}{2} > |L - f(n)| \geq |L| - |f(n)| \\ &\implies |f(n)| \geq |L| - \frac{|L|}{2} = \frac{|L|}{2} > 0 \\ &\implies \left|\frac{1}{f(n)}\right| \leq \frac{2}{|L|}; \end{aligned}$$

i.e., if $M = N_{f-\tilde{L}}\left(\frac{|L|}{2}\right)$, then

$$n \geq M \implies \left|\frac{1}{f(n)}\right| \leq \frac{2}{|L|}.$$

Let

$$B = \max\left(\frac{2}{|L|}, \max_{0 \leq m \leq M}\left|\frac{1}{f(m)}\right|\right).$$

Then $\left|\frac{1}{f(m)}\right| \leq B$ for $m \in \mathbf{Z}_{0 \leq m \leq M}$ and $\left|\frac{1}{f(m)}\right| \leq B$ for $m \in \mathbf{Z}_{\geq M}$, so $\left|\frac{1}{f(m)}\right| \leq B$ for all $m \in \mathbf{Z}_{\geq 0} = \mathbf{N}$, and hence $\frac{1}{f}$ is bounded. |||

7.47 Theorem (Reciprocal theorem for convergent sequences.) *Let g be a complex sequence. Suppose that $g \to L$ where $L \neq 0$, and that $g(n) \neq 0$ for all $n \in \mathbf{N}$. Then $\frac{1}{g}$ is convergent, and $\frac{1}{g} \to \frac{1}{L}$.*

Proof: By the preceding lemma, $\dfrac{1}{g}$ is a bounded sequence, and since $g \to L$, we know that $g - \tilde{L}$ is a null sequence. Hence $(g - \tilde{L}) \cdot \dfrac{1}{g} = \tilde{1} - \dfrac{L}{g}$ is a null sequence, and it follows that $\dfrac{L}{g} \to 1$. Then we have

$$\frac{1}{g} = \frac{1}{L} \cdot \frac{L}{g} \to \frac{1}{L} \cdot 1 = \frac{1}{L};$$

i.e., $\dfrac{1}{g} \to \dfrac{1}{L}$. ‖

7.48 Exercise (Quotient theorem for convergent sequences.) The following statement isn't quite true. Supply the missing hypotheses and prove the corrected statement.

Let f, g be convergent complex sequences. If $f \to L$ and $g \to M$, then $\dfrac{f}{g}$ is convergent and $\dfrac{f}{g} \to \dfrac{L}{M}$.

7.49 Exercise.

a) Let f, g be complex sequences. Show that if f converges and g diverges, then $f + g$ diverges.

b) Show that if f converges and g diverges, then fg does not necessarily diverge.

7.50 Exercise. Let f be a divergent complex sequence. Show that if $c \in \mathbf{C}\backslash\{0\}$, then cf is divergent.

7.51 Example. Let $f\colon \mathbf{Z}_{\geq 1} \to \mathbf{C}$ be defined by

$$f(n) = \frac{n^2 + in + 1}{3n^2 + 2in - 1}. \tag{7.52}$$

Then

$$f(n) = \frac{n^2\left(1 + \frac{i}{n} + \frac{1}{n^2}\right)}{n^2\left(3 + \frac{2i}{n} - \frac{1}{n^2}\right)} = \frac{1 + \frac{i}{n} + \frac{1}{n^2}}{\left(3 - \frac{1}{n^2}\right) + \frac{2i}{n}}. \tag{7.53}$$

Hence f can be written as a quotient of two sequences:

$$h\colon n \mapsto 1 + \frac{i}{n} + \frac{1}{n^2}$$

and

$$g\colon n \mapsto \left(3 - \frac{1}{n^2}\right) + \frac{2i}{n}$$

where $g(n) \neq 0$ for all $n \in \mathbf{Z}_{\geq 1}$. Since

$$h = \tilde{1} + i\left\{\frac{1}{n}\right\}_{n\geq 1} + \left\{\frac{1}{n}\right\}_{n\geq 1} \cdot \left\{\frac{1}{n}\right\}_{n\geq 1}$$

and

$$g = \tilde{3} - \left\{\frac{1}{n}\right\}_{n\geq 1} + 2i\left\{\frac{1}{n}\right\}_{n\geq 1},$$

it follows from numerous applications of product and sum rules that $h \to 1$ and $g \to 3 \neq 0$ and hence $f = \dfrac{h}{g} \to \dfrac{1}{3}$. Once I have expressed $f(n)$ in the final form in (7.53), I can see what the final result is, and I will usually just write

$$\{f(n)\} = \left\{\frac{1 + \frac{i}{n} + \frac{1}{n^2}}{3 - \frac{1}{n^2} + \frac{2i}{n}}\right\} \to \frac{1+0+0}{3-0+0} = \frac{1}{3}.$$

7.54 Example. Let $g\colon \mathbf{N} \to \mathbf{C}$ be the sequence

$$g = \left\{\frac{2^n + 4^n}{4^n + 6^n}\right\}. \tag{7.55}$$

Then for all $n \in \mathbf{N}$,

$$g(n) = \frac{2^n + 4^n}{4^n + 6^n} = \frac{4^n\left(\frac{2^n}{4^n} + 1\right)}{6^n\left(\frac{4^n}{6^n} + 1\right)} = \left(\frac{2}{3}\right)^n \left(\frac{\left(\frac{1}{2}\right)^n + 1}{\left(\frac{2}{3}\right)^n + 1}\right).$$

Since $\left|\dfrac{2}{3}\right| < .7$, I know $\left\{\left(\dfrac{2}{3}\right)^n\right\} \to 0$ and $\left\{\left(\dfrac{1}{2}\right)^n\right\} \to 0$ so

$$\{g(n)\} = \left\{\left(\frac{2}{3}\right)^n \frac{\left(\left(\frac{1}{2}\right)^n + 1\right)}{\left(\left(\frac{2}{3}\right)^n + 1\right)}\right\} \to 0 \cdot \frac{0+1}{0+1} = 0.$$

In the last two examples, I was motivated by the following considerations. I think: In the numerator and denominator for (7.52), for large n the "n^2" term overwhelms the other terms – so that's the term I factored out. In the numerator of (7.55), the overwhelming term is 4^n, and in the denominator, the overwhelming term is 6^n so those are the terms I factored out.

7.56 Exercise. Let $\{f(n)\}$ be a sequence of non-negative numbers and suppose $\{f(n)\} \to L$ where $L > 0$. Prove that $\{\sqrt{f(n)}\} \to \sqrt{L}$. (NOTE: The case $L = 0$ follows from the root theorem for null sequences.

7.57 Exercise. Investigate the sequences below, and find their limits if they have any.

a) $f = \left\{ \dfrac{1 + 3n + 3in^2}{1 + 2in + 5n^2} \right\}_{n \geq 1}$

b) $g = \left\{ \dfrac{n^2 + 3in + 1}{n^3 + n + i} \right\}_{n \geq 1}$

c) $h = \left\{ \dfrac{\left(4 + \frac{1}{n}\right)^2 - 16}{\left(3 + \frac{i}{n}\right)^2 - 9} \right\}_{n \geq 1}$

d) $k = \left\{ \sqrt{1 + \dfrac{1}{n}} \right\}_{n \geq 1}$

e) $l = \left\{ \sqrt{n^2 + n} - n \right\}_{n \geq 1}$

7.58 Exercise. Show that the sum of two bounded sequences is a bounded sequence.

7.59 Theorem (Convergent sequences are bounded.) *Let $\{\alpha_n\}$ be a convergent complex sequence. Then $\{\alpha_n\}$ is bounded.*

Proof: I will show that null sequences are bounded and leave the general case to you. Let f be a null sequence and let N_f be a precision function for f.

Let

$$B = \max\left(1, \max_{0 \leq j \leq N_f(1)} (|f(j)|)\right).$$

I claim that B is a bound for f. If $n \in \mathbf{Z}_{0 \leq j \leq N_f(1)}$, then

$$|f(n)| \leq \max_{0 \leq j \leq N_f(1)} (|f(j)|) \leq B.$$

If $n \in \mathbf{Z}_{\geq N_f(1)}$, then $n \geq N_f(1)$, so $|f(n)| \leq 1 \leq B$. Hence

$$|f(n)| \leq B \text{ for all } n \in \mathbf{Z}_{0 \leq j \leq N_f(1)} \cup \mathbf{Z}_{\geq N_f(1)},$$

i.e., $|f(n)| \leq B$ for all $n \in \mathbf{N}$. ‖

7.60 Exercise. Complete the proof of theorem 7.59; i.e., show that if $\{\alpha_n\}$ is a convergent complex sequence, then $\{\alpha_n\}$ is bounded.

7.61 Example. It follows from the fact that convergent sequences are bounded, that $\{n\}$ is not a convergent sequence.

7.62 Exercise. Give an example of a bounded sequence that is not convergent.

7.6 Geometric Series

7.63 Theorem **($\left\{r^{\frac{1}{n}}\right\} \to 1$.)** *If $r \in \mathbf{R}^+$, then $\{r^{\frac{1}{n}}\} \to 1$.*

Proof:

Case 1: $[r \geq 1]$. By the formula for factoring $s^n - a^n$ (3.78), we have for all $n \in \mathbf{Z}_{\geq 1}$ and all $s \geq 1$

$$(s^n - 1) = (s-1)\sum_{j=0}^{n-1} s^j \geq (s-1)\sum_{j=0}^{n-1} 1^j = n(s-1)$$

so

$$(s-1) \leq \frac{1}{n}(s^n - 1).$$

If we let $s = r^{\frac{1}{n}}$ in this formula, we get

$$|r^{\frac{1}{n}} - 1| = r^{\frac{1}{n}} - 1 \leq \frac{1}{n}(r-1).$$

Since $\left\{\frac{r-1}{n}\right\}$ is a null sequence, it follows from the comparison theorem for null sequences that $\{r^{1/n} - 1\} \to 0$; i.e., $\left\{r^{\frac{1}{n}}\right\} \to 1$.

Case 2: $[0 < r < 1.]$ Let $R = \frac{1}{r}$. Then $R > 1$, so by Case 1, $\left\{R^{\frac{1}{n}}\right\} \to 1$. By the reciprocal theorem $\left\{\frac{1}{R^{\frac{1}{n}}}\right\} \to 1$; i.e., $\left\{r^{\frac{1}{n}}\right\} \to 1$.

We have shown that the theorem holds in all cases. |||

7.64 Theorem (Convergence of geometric sequences.) *Let $\alpha \in \mathbf{C}$. Then*

$$\begin{aligned}&\{\alpha^n\} \to 0 \textit{ if } |\alpha| < 1\\ &\{\alpha^n\} \to 1 \textit{ if } \alpha = 1\\ &\{\alpha^n\} \textit{ diverges if } |\alpha| \geq 1 \textit{ and } \alpha \neq 1.\end{aligned}$$

Proof: The last assertion was shown in theorem 7.7, and the second statement is clear, and it is also clear that $\{\alpha^n\} \to 0$ if $\alpha = 0$.

Suppose that $0 < |\alpha| < 1$. I will show that

$$|\alpha^k| \leq \frac{1}{2} \text{ for some } k \in \mathbf{N}. \tag{7.65}$$

It will then follow that

$$|\alpha^n|^k = (|\alpha|^k)^n \leq \left(\frac{1}{2}\right)^n = \frac{1}{2^n} \text{ for all } n \in \mathbf{N}.$$

Since $\{\frac{1}{2^n}\}$ is a null sequence, it follows from the comparison theorem for null sequences that $\{|\alpha^n|^k\}$ is a null sequence, and then by the root theorem for null sequences (Theorem 7.19), it follows that $\{\alpha^n\}$ is a null sequence.

To prove (7.65), let N be a precision function for $\left\{\left(\frac{1}{2}\right)^{\frac{1}{n}} - 1\right\}$, and let $k = N(1 - |\alpha|)$. Then $\left|\left(\frac{1}{2}\right)^{\frac{1}{k}} - 1\right| < 1 - |\alpha|$, so $1 - \left(\frac{1}{2}\right)^{\frac{1}{k}} < 1 - |\alpha|$, so $|\alpha| < \left(\frac{1}{2}\right)^{\frac{1}{k}}$. and hence $|\alpha^k| \leq \frac{1}{2}$, which is what we wanted to show. |||

7.66 Theorem (Geometric series.) *Let $\alpha \in \mathbf{C}$. If $|\alpha| < 1$, then the geometric series*

$$g_\alpha : n \mapsto \sum_{j=0}^{n} \alpha^j$$

converges to $\frac{1}{1-\alpha}$. If $|\alpha| \geq 1$, then g_α diverges.

Proof: We saw in theorem 3.71 that $g_\alpha(n) = \sum_{j=0}^{n} \alpha^j = \frac{1-\alpha^{n+1}}{1-\alpha}$ for all $\alpha \neq 1$. If $\alpha = 1$, $g_\alpha(n) = n+1$. This sequence diverges, since it is not bounded. If $|\alpha| < 1$, then by the previous theorem $\{\alpha^n\} \to 0$, so

$$\{g_\alpha(n)\} = \left\{\frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha} \cdot \alpha^n\right\} \to \frac{1}{1-\alpha} - \frac{\alpha}{1-\alpha} \cdot 0 = \frac{1}{1-\alpha}.$$

Suppose now $|\alpha| \geq 1$ and $\alpha \neq 1$. Then for all $n \in \mathbf{N}$ we have

$$\begin{aligned} \alpha^n &= \frac{1}{\alpha} \cdot \alpha^{n+1} = \frac{1}{\alpha}\left(1 - (1-\alpha) \cdot \frac{1-\alpha^{n+1}}{1-\alpha}\right) \\ &= \frac{1-(1-\alpha)g_\alpha(n)}{\alpha}. \end{aligned}$$

Hence for all $L \in \mathbf{C}$ we have

$$\{g_\alpha(n)\} \to L \implies \{\alpha^n\} \to \frac{1-(1-\alpha)L}{\alpha}.$$

By theorem 7.7, if $|\alpha| \geq 1$ and $\alpha \neq 1$, then $\{\alpha^n\}$ diverges, and hence $\{g_\alpha(n)\} \to L$ is false for all $L \in \mathbf{C}$; i.e., g_α diverges. ‖

7.67 Notation. If $\{a_j\}_{j\geq 1}$ is a sequence of digits, then we denote $\sum_{j=1}^{n} \frac{a_j}{10^j}$ by $.a_1a_2\cdots a_n$. Thus

$$.14159 = \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5}$$

and

$$\begin{aligned} &.351351351 \\ &= \left(\frac{3}{10} + \frac{5}{100} + \frac{1}{1000}\right) + \left(\frac{3}{10^4} + \frac{5}{10^5} + \frac{1}{10^6}\right) + \left(\frac{3}{10^7} + \frac{5}{10^8} + \frac{1}{10^9}\right) \\ &= \left(\frac{351}{1000}\right)\left[1 + \frac{1}{10^3} + \frac{1}{10^6}\right] \\ &= \frac{351}{1000} \sum_{j=0}^{2} \frac{1}{10^{3j}}. \end{aligned}$$

7.68 Example. Let a, b, c be digits, and let

$$\{x_n\} = \left\{ \frac{abc}{1000} \sum_{j=0}^{n} \frac{1}{10^{3j}} \right\}$$

so informally, $x_n = \underbrace{.abcabc \cdots abc}_{3(n+1) \text{ digits}}$. Then $\{x_n\}$ is a convergent sequence, and

$$\{x_n\} \to \frac{abc}{1000} \cdot \frac{1}{1 - \frac{1}{1000}} = \frac{abc}{999}.$$

As an example, we have

$$\{.351, .351351, .351351351, \cdots\} \to \frac{351}{999} = \frac{39}{111} = \frac{13}{37}.$$

7.69 Exercise. Let

$$\{a_n\} = \{.672, .67272, .6727272, .672727272, \cdots\}_{n \geq 1}.$$

Show that $\{a_n\}$ converges to a rational number.

7.70 Exercise.

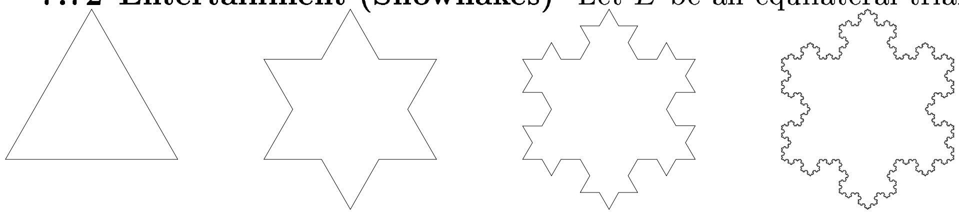
a) Let $\{a_n\} = \left\{ \sum_{j=0}^{n} \left(\left(\frac{3}{5}\right)^j + \left(\frac{4}{5}\right)^j i \right) \right\}$. Does $\{a_n\}$ converge? If it does, find $\lim\{a_n\}$ in the form $a + bi$ where $a, b \in \mathbf{R}$.

b) Let $\{b_n\} = \left\{ \sum_{j=0}^{n} \left(\frac{3+4i}{5}\right)^j \right\}$. Does $\{b_n\}$ converge? If it does, find $\lim\{b_n\}$ in the form $a + bi$ where $a, b \in \mathbf{R}$.

c) Let $\{c_n\} = \left\{ \sum_{j=0}^{n} \left(\left(\frac{3}{5}\right)^j + \left(\frac{4i}{5}\right)^j \right) \right\}$. Does $\{c_n\}$ converge? If it does, find $\lim\{c_n\}$ in the form $a + bi$ where $a, b \in \mathbf{R}$.

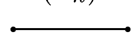
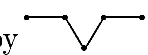
7.71 Exercise. Show that the sequences $\left\{ \sum_{j=0}^{n} \left(\frac{1+i}{2}\right)^j \right\}$ and $\left\{ \sum_{j=0}^{n} \left(\frac{2+i}{3}\right)^j \right\}$ (which are drawn on page 126) converge, and that the limits appear to be in agreement with Figure b) on page 126.

7.72 Entertainment (Snowflakes) Let E be an equilateral triangle with



Snowflakes

area A, and side s. Note that an equilateral triangle with side $\frac{s}{3}$ has area $\frac{A}{9}$. Starting with E, we will now construct a sequence $\{S_n\}$ of polygons. S_n will have $4^n \cdot 3$ sides, all having length $\frac{s}{3^n}$. We let $S_0 = E$ (so S_0 has $4^0 \cdot 3$ sides of length $\frac{s}{3^0}$). To construct S_{n+1} from S_n we attach an equilateral triangle with side of length $\frac{1}{3} \cdot$ side (S_n) to the middle third of each side of S_n.

The bottom side  of S_n will be replaced by . Each side of S_n is replaced by 4 sides of length $\frac{1}{3}\left(\frac{s}{3^n}\right)$, so S_{n+1} will have $4 \cdot (4^n 3) = 4^{n+1} \cdot 3$ sides of length $\frac{s}{3^{n+1}}$. The figure shows some of these polygons. I will call the polygons S_n *snowflake polygons*. We have $S_n \subset S_{n+1}$ for all n. The *snowflake* S is the union of all of the sets S_n; i.e., a point x is in S if and only if it is in S_n for some $n \in \mathbf{N}$.

Find the area of S_n (in terms of the area A of E), for example

$$\text{area}(S_1) = A + 3\left(\frac{A}{9}\right) = \frac{4}{3}A.$$

Then find the area of S in terms of A. Make any reasonable assumptions that you need. What is the perimeter of S?

7.7 The Translation Theorem

7.73 Theorem. *Let f be a real convergent sequence, say $f \to L$. If $f(n) \geq 0$ for all $n \in \mathbf{N}$, then $L \geq 0$.*

Proof: I note that $L \in \mathbf{R}$, since if $f \to L$, then $\text{Re}f \to \text{Re}L$. Suppose, to get a contradiction, that $L < 0$, (so $-\frac{L}{2} > 0$), and let $N_{f-\tilde{L}}$ be a precision function for the null sequence $f-\tilde{L}$. Let $N = N_{f-\tilde{L}}\left(-\frac{L}{2}\right)$. Then $|(f-\tilde{L})(N)| \leq -\frac{L}{2}$, so $|f(N)-L| \leq -\frac{L}{2}$, and hence $f(N) < L - \frac{L}{2} = \frac{L}{2} < 0$. This contradicts the assumption that $f(N) \geq 0$ for all $n \in \mathbf{N}$. $\|$

7.74 Exercise (Inequality theorem.) Let f, g be convergent real sequences. Suppose that $f(n) \leq g(n)$ for all $n \in \mathbf{N}$. Prove that $\lim f \leq \lim g$.

7.75 Exercise. Prove the following assertion, or give an example to show that it is not true. Let f, g be convergent real sequences. Suppose that $f(n) < g(n)$ for all $n \in \mathbf{N}$. Then $\lim f < \lim g$.

7.76 Definition (Translate of a sequence.) Let f be a sequence and let $p \in \mathbf{N}$. Then the sequence $f_p\colon n \mapsto f(n+p)$ is called a *translate of* f.

7.77 Example. If $f = \left\{\frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \cdots, \frac{1}{(n+2)^2}, \cdots\right\}$, then $f_3 = \left\{\frac{1}{5^2}, \frac{1}{6^2}, \frac{1}{7^2}, \cdots, \frac{1}{(n+5)^2}, \cdots\right\}$. A translate of a sequence is a sequence obtained by ignoring the first few terms.

7.78 Theorem (Translation theorem.) *If $\{f(n)\}$ is a convergent complex sequence, and $p \in \mathbf{N}$, then $\{f(n+p)\}$ converges, and $\lim\{f(n)\} = \lim\{f(n+p)\}$. Conversely, if $\{f(n+p)\}$ converges, then $\{f(n)\}$ converges to the same limit.*

Proof: Let $f \to L$, let $f_p(n) = f(n+p)$ and let $N_{f-\tilde{L}}$ be a precision function for $f-\tilde{L}$. I claim $N_{f-\tilde{L}}$ is also a precision function for $f_p - \tilde{L}$. In fact, for all $n \in \mathbf{N}$, and all $\varepsilon \in \mathbf{R}^+$,

$$n \geq N_{f-\tilde{L}}(\varepsilon) \implies n+p \geq N_{f-\tilde{L}}(\varepsilon) \implies |f(n+p) - L| < \varepsilon.$$

Conversely, suppose

$$\{f_p(n)\} = \{f(n+p)\} \to L$$

and let $N_{f_p-\tilde{L}}$ be a precision function for $f_p - \tilde{L}$. Let $N(\varepsilon) = p + N_{f_p-\tilde{L}}(\varepsilon)$ for all $\varepsilon \in \mathbf{R}^+$. I claim N is a precision function for $N_{f-\tilde{L}}$. For all $n \in \mathbf{N}$,

$$\begin{aligned} n > N(\varepsilon) &\implies n > p + N_{f_p-\tilde{L}}(\varepsilon) \\ &\implies n - p > N_{f_p-\tilde{L}}(\varepsilon) \\ &\implies |(f_p - \tilde{L})(n-p)| < \varepsilon \\ &\implies |f_p(n-p) - L| < \varepsilon \\ &\implies |f(n) - L| < \varepsilon. \ \| \end{aligned}$$

7.79 Example. Let the sequence f be defined by

$$\begin{aligned} f(0) &= 1, \\ f(n+1) &= \frac{1}{1+f(n)} \text{ for all } n \in \mathbf{N}. \end{aligned}$$

Then

$$\begin{aligned} f(1) &= \frac{1}{1+1} = \frac{1}{2} \\ f(2) &= \frac{1}{1+\frac{1}{1+1}} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3} \\ f(3) &= \frac{1}{1+\frac{1}{1+\frac{1}{1+1}}} = \frac{1}{1+\frac{2}{3}} = \frac{3}{5}. \end{aligned}$$

Suppose I knew that f converged to a limit L. It is clear that $f(n) > 0$ for all n, so L must be ≥ 0. By the translation theorem

$$L = \lim\{f(n+1)\} = \lim\left\{\frac{1}{1+f(n)}\right\} = \frac{1}{1+\lim f(n)} = \frac{1}{1+L}$$

so $L(1+L) = 1$; i.e., $L^2+L-1 = 0$. Hence $L \in \left\{\frac{-1+\sqrt{1+4}}{2}, \frac{-1-\sqrt{1+4}}{2}\right\}$, and since $L \geq 0$, we conclude $L = \frac{\sqrt{5}-1}{2}$. I've shown that the only thing that f can possibly converge to is $\frac{\sqrt{5}-1}{2}$. Now

$$0 < L < \frac{3-1}{2} = 1, \quad \text{so} \quad |1 - L| < 1.$$

Since $L = \frac{1}{1+L}$, we have for all $n \in \mathbf{N}$,

$$\begin{aligned} |f(n+1) - L| &= \left| \frac{1}{1+f(n)} - \frac{1}{1+L} \right| = \left| \frac{L - f(n)}{(1+f(n))(1+L)} \right| \leq \frac{|L - f(n)|}{1+L} \\ &= L|L - f(n)| = L|f(n) - L|. \end{aligned}$$

Hence

$$\begin{aligned} |f(1) - L| &\leq L|f(0) - L| = L|1 - L| \leq L, \\ |f(2) - L| &\leq L|f(1) - L| \leq L^2, \\ |f(3) - L| &\leq L|f(2) - L| \leq L^3, \end{aligned}$$

and by induction,

$$|f(n) - L| \leq L^n \text{ for all } n \in \mathbf{Z}_{\geq 1}.$$

By theorem 7.64 $\{L^n\}$ is a null sequence, and by the comparison theorem for null sequences, it follows that $\{f(n) - L\}$ is a null sequence. This completes the proof that $f \to L$. |||

7.80 Exercise. Let

$$\begin{aligned} f(0) &= -2 \\ f(n+1) &= \frac{f(n)^2 + 2}{2f(n)} \text{ for all } n \in \mathbf{N}. \end{aligned}$$

a) Assume that f converges, and determine the value of $\lim\{f(n)\}$.

b) Calculate $f(1), f(2), f(3), f(4)$, using all of the accuracy of your calculator. Does the sequence appear to converge?

7.81 Entertainment. Show that the sequence f defined in the previous exercise converges. We will prove this result in Example 7.97, but you can prove it now, using results you know.

7.82 Exercise. Let g be the sequence defined by

$$\begin{aligned} g(0) &= 1, \\ g(1) &= 1, \\ g(n+2) &= \frac{1 + g(n+1)}{g(n)} \text{ for all } n \in \mathbf{N}. \end{aligned}$$

a) Assume that g converges, and determine the value of $\lim\{g(n)\}$.

b) Calculate $g(1), g(2), g(3), g(4), g(5), g(6)$, using all of the accuracy of your calculator. Does this sequence converge?

7.83 Theorem (Divergence test.) *Let f, g be complex sequences such that $g(n) \neq 0$ for all $n \in \mathbf{N}$. Suppose that $g \to 0$ and $f \to L$ where $L \neq 0$. Then $\frac{f}{g}$ diverges.*

Proof: Suppose, to get a contradiction, that $\frac{f}{g}$ converges to a limit M. Then by the product theorem, $g \cdot \frac{f}{g}$ converges to $0 \cdot M = 0$; i.e., $f \to 0$. This contradicts our assumption that f has a non-zero limit. |||

7.84 Exercise. Prove the following assertion or give an example to show that it is not true: Let f, g be complex sequences such that $g(n) \neq 0$ for all $n \in \mathbf{N}$, but $g \to 0$. Then $\frac{f}{g}$ diverges.

7.85 Example. Let $f(n) = \left\{\frac{n^3 + 3n}{n^2 + 1}\right\}$ for all $n \in \mathbf{Z}_{\geq 1}$. Then

$$f(n) = \frac{n^3\left(1 + \frac{3}{n^2}\right)}{n^2\left(1 + \frac{1}{n}\right)} = \frac{\left(1 + \frac{3}{n^2}\right)}{\frac{1}{n}\left(1 + \frac{1}{n}\right)}.$$

Since

$$\lim\left\{\left(1 + \frac{3}{n^2}\right)\right\}_{n \geq 1} = 1 + 0 \neq 0,$$

and

$$\lim\left\{\frac{1}{n}\left(1 + \frac{1}{n}\right)\right\}_{n \geq 1} = 0 \cdot (1 + 0) = 0,$$

it follows that f diverges.

7.86 Exercise. Let A, B, a, b be complex numbers such that $an + b \neq 0$ for all $n \in \mathbf{Z}_{\geq 1}$. Discuss the convergence of $\left\{\frac{An + B}{an + b}\right\}_{n \geq 1}$. Consider all possible choices for A, B, a, b.

7.8 Bounded Monotonic Sequences

7.87 Theorem. *Let $\{[a_n, b_n]\}$ be a binary search sequence in* $\mathbf{R}$. *Suppose $\{[a_n, b_n]\} \to c$ where $c \in \mathbf{R}$. Then $\{b_n - a_n\}$ is a null sequence. Also $\{a_n\} \to c$ and $\{b_n\} \to c$.*

Proof: We know that $b_n - a_n = \dfrac{b_0 - a_0}{2^n}$, and that $\{\frac{1}{2^n}\}$ is a null sequence, so $\{b_n - a_n\}$ is a null sequence. Since $\{[a_n, b_n]\} \to c$ we know that $a_n \leq c \leq b_n$ for all $n \in \mathbf{N}$, and hence

$$0 \leq |b_n - c| \leq |b_n - a_n| \qquad \text{and} \qquad 0 \leq |c - a_n| \leq |b_n - a_n|$$

for all $n \in \mathbf{N}$. By the comparison theorem for null sequences it follows that $\{c - a_n\}$ and $\{b_n - c\}$ are null sequences, and hence $\{a_n\} \to c$ and $\{b_n\} \to c$. ‖

7.88 Definition (Increasing, decreasing, monotonic) Let f be a real sequence. We say f is *increasing* if $f(n) \leq f(n+1)$ for all $n \in \mathbf{N}$, and we say f is *decreasing* if $f(n) \geq f(n+1)$ for all $n \in \mathbf{N}$. We say that f is *monotonic* if either f is increasing or f is decreasing.

7.89 Theorem. Let f be an increasing real sequence. Then for all $k, n \in \mathbf{N}$

$$f(k) \leq f(k+n).$$

Proof: Define a proposition form P on $\mathbf{N}$ by

$$P(n) = \text{"for all } k \in \mathbf{N}(f(k) \leq f(k+n))\text{"}, \text{ for all } n \in \mathbf{N}.$$

Then $P(0)$ says "for all $k \in \mathbf{N}(f(k) \leq f(k))$", so $P(0)$ is true. Since f is increasing, we have for all $n \in \mathbf{N}$,

$$\begin{aligned} P(n) &\implies \text{for all } k \in \mathbf{N}\Big(f(k) \leq f(k+n) \leq f((k+n)+1)\Big) \\ &\implies \text{for all } k \in \mathbf{N}\Big(f(k) \leq f(k+(n+1))\Big) \\ &\implies P(n+1). \end{aligned}$$

By induction, we conclude that $P(n)$ is true for all $n \in \mathbf{N}$, i.e.

$$\text{for all } n \in \mathbf{N}\Big(\text{for all } k \in \mathbf{N}(f(k) \leq f(k+n))\Big). \ ‖$$

7.90 Corollary. Let f be an increasing real sequence. Then for all $k, n \in \mathbf{N}$,

$$k \leq n \implies f(k) \leq f(n). \tag{7.91}$$

Proof: For all $k, n \in \mathbf{N}$

$$k \leq n \implies n - k \in \mathbf{N} \implies f(k) \leq f(k + (n - k)) = f(n). \parallel$$

7.92 Definition (Upper bound, lower bound.) Let f be a real sequence. We say that f has an *upper bound* if there is a number $U \in \mathbf{R}$ such that $f(n) \leq U$ for all $n \in \mathbf{N}$. Any such number U is called an *upper bound for* f. We say that f has a *lower bound* if there is a number $L \in \mathbf{R}$ such that $L \leq f(n)$ for all $n \in \mathbf{N}$. Any such number L is called a *lower bound for* f.

7.93 Examples. If $f(n) = \frac{(-1)^n n}{n+1}$ for all $n \in \mathbf{N}$ then 1 (or any number greater than 1) is an upper bound for f, and -1 (or any number less than -1) is a lower bound for f. The sequence $g : n \mapsto n$ has no upper bound, but 0 is a lower bound for g.

7.94 Exercise. In definition 7.41, we defined a complex sequence f to bounded if there is a number $B \in [0, \infty)$ such that $|f(n)| \leq B$ for all $n \in \mathbf{N}$. Show that a real sequence is bounded if and only if it has both an upper bound and a lower bound.

7.95 Theorem (Bounded monotonic sequences converge.) *Let f be an increasing sequence in $\mathbf{R}$, and suppose f has an upper bound. Then f converges. (Similarly, decreasing sequences that have lower bounds converge.)*

Proof: Let B be an upper bound for f. We will construct a binary search sequence $\{[a_n, b_n]\}$ satisfying the following two conditions:

i. For every $n \in \mathbf{N}$, b_n is an upper bound for f,

ii. For every $n \in \mathbf{N}$, a_n is not an upper bound for f.

Let

$$\begin{aligned} [a_0, b_0] &= [f(0) - 1, B] \\ [a_{n+1}, b_{n+1}] &= \begin{cases} \left[a_n, \frac{a_n + b_n}{2}\right] & \text{if } \frac{a_n + b_n}{2} \text{ is an upper bound for } f \\ \left[\frac{a_n + b_n}{2}, b_n\right] & \text{if } \frac{a_n + b_n}{2} \text{ is not an upper bound for } f. \end{cases} \end{aligned}$$

A straightforward induction argument shows that $\{[a_n, b_n]\}$ satisfies conditions i) and ii).

Let c be the number such that $\{[a_n, b_n]\} \to c$. I will show that $f \to c$.

We know that $\{b_n - a_n\} = \{\frac{b_0 - a_0}{2^n}\}$ is a null sequence. Let N be a precision function for $\{b_n - a_n\}$, so that for all $\varepsilon \in \mathbf{R}^+$,

$$n \geq N(\varepsilon) \implies |b_n - a_n| < \varepsilon.$$

I will use N to construct a precision function K for $f - \tilde{c}$.

Let $\varepsilon \in \mathbf{R}^+$. Since $a_{N(\varepsilon)}$ is not an upper bound for f, there is a number $K(\varepsilon) \in \mathbf{N}$ such that $f(K(\varepsilon)) > a_{N(\varepsilon)}$. By condition i), I know that $f(n) \leq b_{N(\varepsilon)}$ for all $n \in \mathbf{N}$. Hence, since f is increasing, we have for all $n \in \mathbf{N}$:

$$\begin{aligned} n \geq K(\varepsilon) &\implies a_{N(\varepsilon)} < f(K(\varepsilon)) \leq f(n) \leq b_{N(\varepsilon)} \\ &\implies f(n) \in [a_{N(\varepsilon)}, b_{N(\varepsilon)}]. \end{aligned}$$

Since $\{[a_n, b_n]\} \to c$ we also have

$$c \in [a_{N(\varepsilon)}, b_{N(\varepsilon)}].$$

Hence

$$|f(n) - c| \leq b_{N(\varepsilon)} - a_{N(\varepsilon)} < \varepsilon \text{ for all } n \geq K(\varepsilon).$$

This says that K is a precision function for $\{f(n) - c\}$, and hence $f \to c$ |||

7.96 Corollary. *Let f be a real sequence. If f has an upper bound, and there is some $N \in \mathbf{N}$ such that*

$$f(n+1) \geq f(n) \text{ for all } n \in \mathbf{Z}_{\geq N}$$

then f converges. Similarly, if f has a lower bound, and there is some $N \in \mathbf{N}$ such that

$$f(n+1) \leq f(n) \text{ for all } n \in \mathbf{Z}_{\geq N}$$

then f converges.

7.97 Example. Let $a \in \mathbf{R}^+$. Define a sequence $\{x_n\}$ by

$$\begin{aligned} x_0 &= a + 1 \\ x_{n+1} &= \frac{x_n^2 + a}{2x_n} \text{ for all } n \in \mathbf{N}. \end{aligned}$$

We have $x_n > 0$ for all n. Suppose $\{x_n\}$ converges to a limit L. Since $2x_n x_{n+1} = x_n^2 + a$ for all $n \in \mathbf{N}$, we can use the translation theorem to show that

$$2L^2 = 2\lim\{x_n\}\lim\{x_{n+1}\} = \lim\{x_n^2 + a\} = L^2 + a,$$

so $2L^2 = L^2 + a$, and hence $L^2 = a$, so L must be $\pm\sqrt{a}$. Since $x_n \geq 0$ for all n, it follows from the inequality theorem that $L \geq 0$, and hence if $\{x_n\}$ converges, it must converge to $\sqrt{a}$. In order to show that $\{x_n\}$ converges, it is sufficient to show that $\{x_n\}$ is decreasing. (We've already noted that 0 is a lower bound.)

Well,

$$x_n - x_{n+1} = x_n - \frac{x_n^2 + a}{2x_n} = \frac{2x_n^2 - x_n^2 - a}{2x_n} = \frac{x_n^2 - a}{2x_n},$$

so if I can show that $x_n^2 - a \geq 0$ for all $n \in \mathbf{N}$, then I'll know that $\{x_n\}$ is decreasing. Now

$$\begin{aligned} x_{n+1}^2 - a &= \left(\frac{x_n^2 + a}{2x_n}\right)^2 - a = \frac{x_n^4 + 2ax_n^2 + a^2}{4x_n^2} - a \\ &= \frac{x_n^4 + 2ax_n^2 + a^2 - 4ax_n^2}{4x_n^2} = \frac{(x_n^2 - a)^2}{4x_n^2} \geq 0. \end{aligned}$$

I also note that $x_0^2 - a = a^2 + a + 1 > 0$, so I finally conclude that $\{x_n\}$ is decreasing, and hence $\{x_n\} \to \sqrt{a}$. In fact, this sequence converges very fast, and is the basis for the square root algorithm used on most computers.

7.98 Example **($\{n^{\frac{1}{n}}\}$)** We will show that $\{n^{\frac{1}{n}}\}_{n\geq 1} \to 1$.
Claim: $\{n^{\frac{1}{n}}\}_{n\geq 3}$ is a decreasing sequence.
Proof: For all $n \in \mathbf{Z}_{\geq 1}$,

$$\begin{aligned} (n+1)^{\frac{1}{n+1}} \leq n^{\frac{1}{n}} &\iff (n+1)^n \leq n^{n+1} \\ &\iff \left(\frac{n+1}{n}\right)^n \leq n. \end{aligned} \tag{7.99}$$

We will show by induction that (7.99) holds for all $n \in \mathbf{Z}_{\geq 3}$. Let

$$P(n) = \text{“}\left(\frac{n+1}{n}\right)^n \leq n\text{” for all } n \in \mathbf{Z}_{\geq 3}.$$

Then $P(3)$ says $(\frac{4}{3})^3 \le 3$, which is true since $64 < 81$. For all $n \in \mathbf{Z}_{\ge 3}$,

$$\begin{aligned}
P(n) &\implies \left(\frac{n+1}{n}\right)^n \le n \\
&\implies \left(\frac{n+2}{n+1}\right)^{n+1} = \left(\frac{n+1}{n}\right)^{n+1} \left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1} \\
&\qquad = \left(\frac{n+1}{n}\right)^n \cdot \frac{n+1}{n} \left(\frac{n^2+2n}{n^2+2n+1}\right)^{n+1} \le n \cdot \frac{n+1}{n} \cdot 1 \\
&\implies \left(\frac{n+2}{n+1}\right)^{n+1} \le n+1 \\
&\implies P(n+1).
\end{aligned}$$

By induction, $P(n)$ is true for all $n \in \mathbf{Z}_{\ge 3}$, and the claim is proved.

Let $L = \lim\{n^{\frac{1}{n}}\}$. Then $\{(2n)^{\frac{1}{2n}}\} \to L$, since any precision function for $\{n^{\frac{1}{n}}\}$ is also a precision function for $\{(2n)^{\frac{1}{2n}}\}$. Hence

$$L^2 = \lim\{\left((2n)^{\frac{1}{2n}}\right)^2\} = \lim\{2^{\frac{1}{n}} n^{\frac{1}{n}}\} = 1 \cdot L = L.$$

Thus $L^2 = L$, and hence $L \in \{0, 1\}$. Since $n^{\frac{1}{n}} \ge 1$ for all $n \in \mathbf{Z}_{\ge 3}$ it follows from the inequality theorem that $L \ge 1$, and hence $L = 1$. ‖

7.100 Exercise. Show that the sequence

$$\left\{\frac{60^n}{n!}\right\} = \{1, 60, 1800, 36000, \cdots\}$$

is a null sequence.

7.101 Exercise. Criticize the following argument.

We know that $\left\{1 + \frac{1}{n}\right\}_{n \ge 1} \to 1 + 0 = 1$.

Hence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}_{n \ge 1} \to 1^n = 1$. ‖

7.102 Note. I got the idea of using precision functions from a letter by Jan Mycielski in the *Notices of the American Mathematical Society*[34, p 569]. Mycielski calls precision functions *Skolem functions*.

The snowflake was introduced by Helge von Koch(1870–1924) who published his results in 1906 [32]. Koch considered only the part of the boundary corresponding to the bottom third of our polygon, which he introduced as an example of a curve not having a tangent at any point.

The sequence g from Exercise 7.82 is taken from [12, page 55, ex 20]