Chapter 6

The Complex Numbers

Many of the results in this chapter are informal and geometrical, and do not follow logically from our assumptions. I will freely use properties of similar triangles, parallelograms, and trigonometric functions. Some of the results (e.g., those involving trigonometric identities) will be rederived later in a more rigorous form. Every statement labeled **Theorem** or **Definition** is part of our logical development.

6.1 Absolute Value and Complex Conjugate

6.1 Definition (Complex Numbers, C.) We denote the complexification of $\mathbf{R}$ by $\mathbf{C}$, and we call $\mathbf{C}$ the *complex numbers*.

6.2 Definition (Absolute value.) In exercise 4.23 we showed that (for any field F in which -1 is not a square), if $z = a + bi = (a, b) \in \mathbf{C}_F$, then

$$z^* z = a^2 + b^2 \in F.$$

If we are working in $\mathbf{C}$, then $a^2 + b^2 \in [0, \infty)$ and hence zz^* has a unique square root in $[0, \infty)$, which we denote by $|z|$ and call the *absolute value* of z.

$$|z| = (z^* z)^{1/2} \text{ for all } z \in \mathbf{C}.$$

We note that

$$|z| \in \mathbf{R}^+ \cup \{0\} \text{ for all } z \in \mathbf{C}.$$
$$|z| = 0 \iff z = 0.$$

Also note that for $z \in \mathbf{R}$, this definition agrees with our old definition of absolute value in $\mathbf{R}$.

6.3 Definition (Real and imaginary parts.) Let $z \in \mathbf{C}$ and write $z = x + iy$ where $x, y \in \mathbf{R}$. We call x the *real part of* z, and we call y the *imaginary part of* z (note that the imaginary part of z is real), and we write

$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z) \text{ if } z = (x, y) = x + iy.$$

6.4 Theorem. *Let z, w be complex numbers. Then*

a) $|zw| = |z|\,|w|$.

b) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ *if* $w \neq 0$.

c) $\mathrm{Re}(z) = \dfrac{z + z^*}{2}$.

d) $\mathrm{Im}(z) = \dfrac{z - z^*}{2i}$.

e) $|\mathrm{Re}(z)| \leq |z|$.

f) $|\mathrm{Im}(z)| \leq |z|$.

g) $|z^*| = |z|$.

h) $\mathrm{Re}(z + w) = \mathrm{Re}(z) + \mathrm{Re}(w)$.

i) $\mathrm{Im}(z + w) = \mathrm{Im}(z) + \mathrm{Im}(w)$.

Proof: By using properties of the complex conjugate proved in exercise 4.23, we have

$$|zw|^2 = (zw)^*(zw) = z^*w^*zw = z^*zw^*w = |z|^2|w|^2.$$

Hence by uniqueness of square roots, $|zw| = |z|\,|w|$. The proofs of b), c), d), e), f), g), h) and i) are left to you. |||

6.5 Exercise. Prove parts b), c), d), e), f), g), h) and i) of Theorem 6.4.

6.6 Theorem (Triangle inequality.) *Let $z, w \in \mathbf{C}$. Then*

$$|z+w| \leq |z| + |w|.$$

Proof: For all $z, w \in \mathbf{C}$,

$$\begin{aligned} |z+w|^2 &= (z+w)^* \cdot (z+w) = (z^* + w^*) \cdot (z+w) \\ &= z^* z + z^* w + w^* z + w^* w \\ &= |z|^2 + z^* w + w^* z + |w|^2. \end{aligned} \tag{6.7}$$

Now since $z^{**} = z$, we have

$$\begin{aligned} z^* w + w^* z &= (z^* w) + (z^* w)^* \\ &= 2\mathrm{Re}(z^* w) \leq 2|\mathrm{Re}(z^* w)| \\ &\leq 2|z^* w| = 2|z^*|\ |w| = 2|z|\ |w|. \end{aligned}$$

Hence, from (6.7),

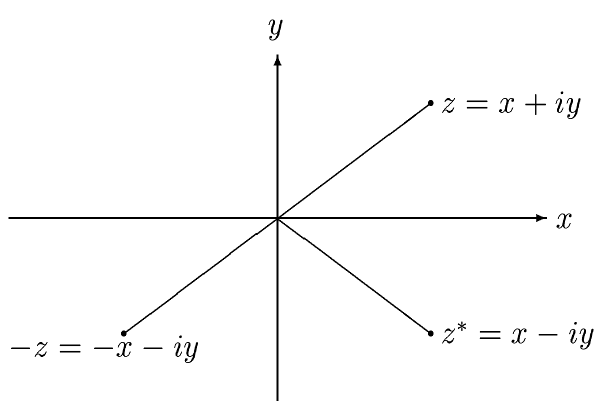
$$|z+w|^2 \leq |z|^2 + 2|z|\ |w| + |w|^2 = (|z| + |w|)^2,$$

and it follows that

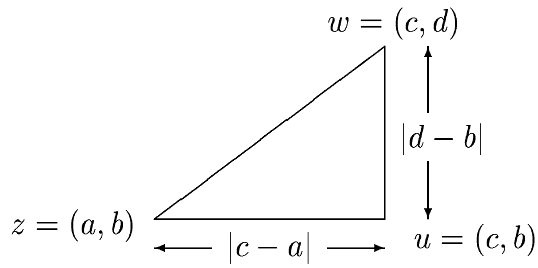
$$|z+w| \leq |z| + |w|. \ \|$$

6.2 Geometrical Representation

Since $\mathbf{C} = \mathbf{R} \times \mathbf{R}$, we can identify complex numbers with points in a plane.



Then $\mathbf{R}$ is identified with the x-axis, and points on the y-axis are of the form iy where y is real. I will call the x-axis the *real axis*, and I'll call the y-axis the *imaginary axis*. If $z \in \mathbf{C}$, then z^* represents the result of reflecting z about the real axis. Also $-z$ represents the result of reflecting z through the origin.



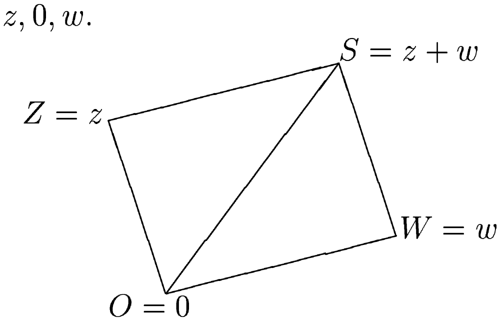
If $z = (a, b)$ and $w = (c, d)$ are two points in $\mathbf{C}$, and $u = (c, b)$, then z, u, w are the vertices of a right triangle having legs of length $|c - a|$, and $|d - b|$. By the Pythagorean theorem, the distance from w to z is $\sqrt{(c-a)^2 + (d-b)^2}$. Also,

$$\begin{aligned} |w - z| &= |(c + id) - (a + ib)| \\ &= |(c - a) + i(d - b)| \\ &= \sqrt{(c-a)^2 + (d-b)^2} \\ &= \text{ distance from } w \text{ to } z, \end{aligned}$$

and in particular, for $z = 0$,

$$|w| = \text{ distance from } w \text{ to } 0.$$

Claim: If $z, w \in \mathbf{C}$, then $z + w$ is the fourth vertex of the parallelogram having consecutive vertices $z, 0, w$.



To make this look like a geometry proof, I'll denote points by upper case letters, and let AB denote the distance from A to B. Let $O = 0$, $W = w$, $Z = z$, $S = z + w$. Then

$$ZS = |(z + w) - z| = |w| = |w - 0| = OW$$

$$WS = |(z + w) - w| = |z| = |z - 0| = OZ.$$

Hence, since the quadrilateral $OWSZ$ has opposite sides equal, it is a parallelogram.

We can now give a geometrical interpretation for the triangle inequality (which motivates its name). In the figure above,

$$|z + w| \leq |z| + |w|$$

says

$$OS \leq OZ + ZS;$$

i.e, the sum of two sides of a triangle is greater than or equal to the third side. This is proposition 20 of book 1 of Euclid [19] "In any triangle, two sides taken together in any manner are greater than the remaining one." (Euclid did not consider triangles in which all three vertices lie on a line.)

> It was the habit of the Epicureans, says Proclus ... to ridicule this theorem as being evident even to an ass, and requiring no proof, and their allegation that the theorem was "known" ($\gamma\nu\acute{\omega}\rho\iota\mu o\nu$) even to an ass was based on the fact that, if fodder is placed at one angular point and the ass at another, he does not, in order to get his food, traverse the two sides of the triangle but only the one side separating them [19, vol. I page 287].

6.8 Definition (Circle, disc.) Let $\alpha \in \mathbf{C}, r \in \mathbf{R}^+$. The *circle* with center α and radius r is

$$\begin{aligned} C(\alpha, r) &= \{z \in \mathbf{C}\colon |z - \alpha| = r\} \\ &= \text{set of points whose distance from } a \text{ is } r. \end{aligned}$$

The *open disc* with center α and radius r is

$$D(\alpha, r) = \{z \in \mathbf{C}\colon |z - \alpha| < r\},$$

and the *closed disc* with center α and radius r is

$$\bar{D}(\alpha, r) = \{z \in \mathbf{C}\colon |z - \alpha| \leq r\}.$$

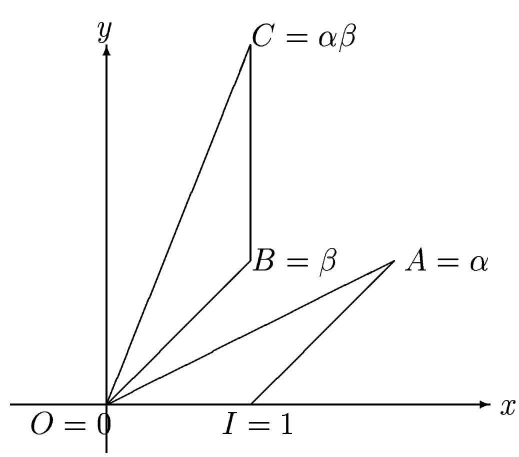
$C(0,1)$ is called the *unit circle*, and $D(0,1)$ is called the *unit disc*. A complex number z is in the unit circle if and only if $|z| = 1$.

6.9 Warning. The word "circle" is sometimes used to mean "disc", although the word "disc" is never used to mean "circle". When you see the word "circle" used in a mathematical statement, you should determine from the context which of the two words is meant. For example, in the statement "the area of the unit circle is π", the word "circle" means "disc", since the unit circle is, in fact, a zero-area set. In these notes the word "circle" always means "circle" except on page 92.

6.10 Theorem. *The product of two numbers in the unit circle is in the unit circle.*

Proof: Let $\alpha, \beta \in C(0,1)$; i.e., $|\alpha| = |\beta| = 1$. Then $|\alpha\beta| = |\alpha||\beta| = 1 \cdot 1 = 1$, so $\alpha\beta \in C(0,1)$. ‖

We can also give a geometrical interpretation to the product of two complex numbers. Let $\alpha = A$ and $\beta = B$ be complex numbers and let $C = \alpha\beta$. Let $O = 0$ and let $I = 1$.

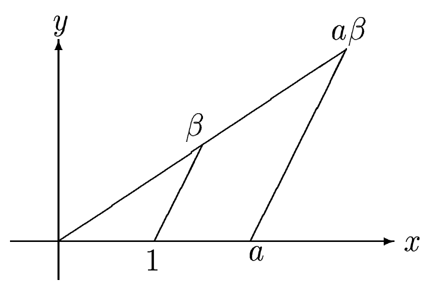


Then $\triangle OIA$ is similar to $\triangle OBC$. The proof consists in showing that

$$\frac{OI}{OB} = \frac{IA}{BC} = \frac{OA}{OC}. \tag{6.11}$$

6.12 Exercise. Prove the equalities listed in (6.11). Assume $\alpha \notin \{0, 1\}$ and $\beta \neq 0$.

From the similarity of $\triangle OIA$ and $\triangle OBC$, we have $\angle IOA = \angle BOC$. In particular, if we take $\alpha = a \in \mathbf{R}^+$, we get the picture



where

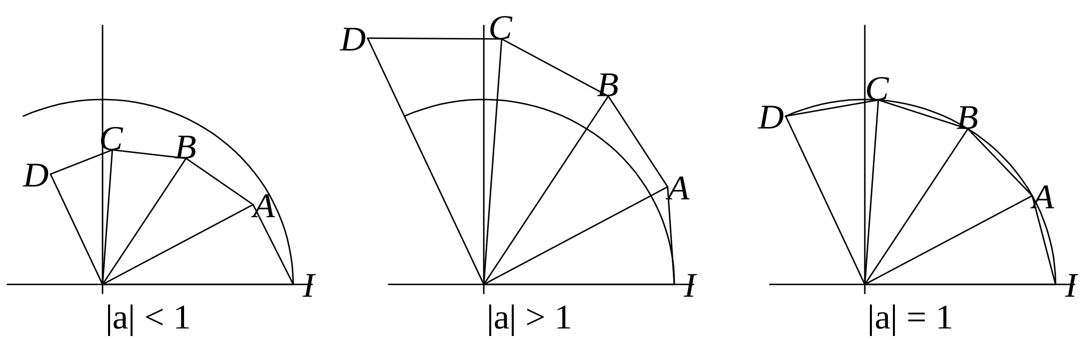
angle(1-0-a)= angle (β-0-$a\beta$),

which indicates that $a\beta$ lies on the line through 0 that passes through β. Also

$$|a\beta| = |a|\ |\beta| = a|\beta|$$

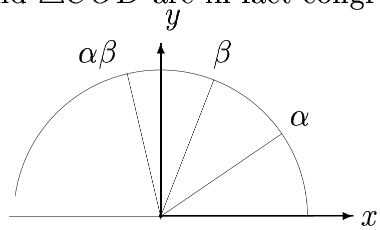
so the length of $a\beta$ is obtained by multiplying the length of β by a.

The figure below shows the powers of a complex number a.



Powers of a: $I = 1$, $A =$ a, $B =$a^2, $C =$ a^3, $D =$ a^4.

In each case the four triangles $\triangle IOA$, $\triangle AOB$, $\triangle BOC$, and $\triangle COD$ are all similar. In the third figure, where a is in the unit circle, the triangles $\triangle IOA, \triangle AOB, \triangle BOC$ and $\triangle COD$ are in fact congruent.



If α, β are points on the unit circle, then

$$\text{angle}(\beta\text{-}0\text{-}\alpha\beta) = \text{angle}(1\text{-}0\text{-}\alpha),$$

which indicates that $\alpha\beta$ is the point in the unit circle such that

$$\text{angle}(1\text{-}0\text{-}\alpha\beta) = \text{angle}(1\text{-}0\text{-}\alpha) + \text{angle}(1\text{-}0\text{-}\beta).$$

From trigonometry, you know that the point on the unit circle making angle θ with the segment OI is $(\cos\theta, \sin\theta) = \cos\theta + i\sin\theta$.

The previous geometrical argument suggests that

$$(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = (\cos(\theta+\phi) + i\sin(\theta+\phi)). \tag{6.13}$$

6.14 Exercise. Using standard trigonometric identities, prove (6.13), and show that $(\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$ for all $\theta \in \mathbf{R}$.

6.15 Exercise. Let $\theta \in \mathbf{R}$. Let $n \in \mathbf{N}$. Prove that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta). \tag{6.16}$$

Then show that formula (6.16) is in fact valid for all $n \in \mathbf{Z}$. (Formula (6.16) is called *De Moivre's Formula.*)

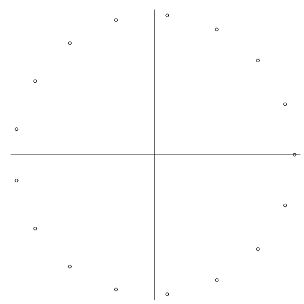
6.3 Roots of Complex Numbers

I expect from (6.16) that every point $(\cos\theta, \sin\theta)$ in the unit circle has nth roots for all $n \in \mathbf{Z}_{\geq 1}$, and that in fact

$$\left(\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right)^n = \cos\theta + i\sin\theta.$$

In particular, each vertex of the regular n-gon inscribed in the unit circle and having a vertex at 1 will be an nth root of 1.

6.17 Exercise. The figure below shows the seventeen points
$\left\{\left(\cos\frac{2\pi j}{17}+i\sin\frac{2\pi j}{17}\right):0\leq j<17\right\}.$



Let $w=\left(\cos\frac{4\pi}{17}+i\sin\frac{4\pi}{17}\right)$ and $u=\left(\cos\frac{10\pi}{17}+i\sin\frac{10\pi}{17}\right)$. Draw the polygons 1-$w$-$w^2$-$\cdots$-$w^{17}$ and 1-u-u^2-$\cdots$-u^{17} on different sets of axes, (i.e. draw segments connecting 1 to w, w to w^2, $\cdots$, w^{16} to w^{17}, and segnents joining 1 to u, $\cdots$, u^{16} to u^{17}.)

6.18 Exercise. The sixth roots of 1 are the vertices of a regular hexagon having one vertex at 1. Find these numbers (by geometry or trigonometry) in terms of rational numbers or square roots of rational numbers, and verify by direct calculation that all of them do, in fact, have sixth power equal to 1.

6.19 Theorem (Polar decomposition.) *Let $z\in\mathbf{C}\backslash\{0\}$. Then we can write $z=ru$ where $r\in\mathbf{R}^+$ and $u\in C(0,1)$. In fact this representation is unique, and*

$$r=|z|,\qquad u=\frac{z}{|z|}.$$

I will call the representation

$$z=ru \text{ where } r\in\mathbf{R}^+,\ u\in C(0,1)$$

the polar decomposition *of z, and I'll call r the* length of z, *and I'll call u the* direction of z.

Proof: If $z=ru$ where $r\in\mathbf{R}^+$ and $|u|=1$, then we have

$$|z|=|ru|=|r|\,|u|=r\cdot 1=r.$$

This shows that $r = |z|$, and it then follows that $u = \frac{z}{r} = \frac{z}{|z|}$. Since $\left|\frac{z}{|z|}\right| = \frac{|z|}{||z||} = 1$, we see $\frac{z}{|z|} \in C(0,1)$ and $z = |z|\left(\frac{z}{|z|}\right)$ gives the desired decomposition. ‖

6.20 Notation (Direction.) I will refer to any number in $C(0,1)$ as a *direction.*

6.21 Example. The polar decomposition for $-1+i$ is

$$\begin{aligned}(-1+i) &= |-1+i|\left(\frac{-1+i}{|-1+i|}\right)\\ &= \sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right).\end{aligned}$$

I recognize from trigonometry that $\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) = \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$.

6.22 Remark. Let $z, w \in \mathbf{C}\backslash\{0\}$. Let $z = ru$ and $w = sv$ be the polar decompositions of z, w, respectively, so $r, s \in \mathbf{R}^+; u, v \in C(0,1)$. Then $zw = rusv = (rs)(uv)$ where $rs \in \mathbf{R}^+$ and $uv \in C(0,1)$. Hence we have

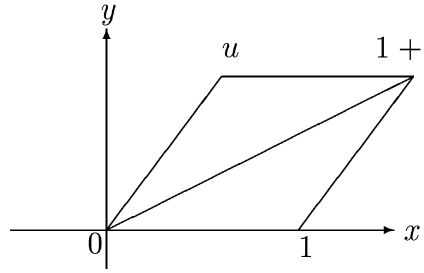
length of product = product of lengths

and

direction of product = product of directions.

6.4 Square Roots

Let u be a direction in $C(0,1)$, with $u \neq -1$. Then we know that $1, 0, u, 1+u$ are the vertices of a parallelogram.



Since $|u| = |1| = 1$, all four sides of the parallelogram are equal, and thus the parallelogram is a rhombus. Since the diagonals of a rhombus bisect its angles, the segment from 0 to $1+u$ bisects angle (1-0-u). Hence I expect that the direction of $1+u$ (i.e., $\dfrac{1+u}{|1+u|}$) is a square root of u. I can prove that this is the case without using any geometry.

6.23 Theorem. *Let u be a direction in $\mathbf{C}$ with $u \neq -1$. Then $\dfrac{1+u}{|1+u|}$ is a square root of u.*

Proof: I just need to square $\dfrac{1+u}{|1+u|}$. Well,

$$\left(\frac{1+u}{|1+u|}\right)^2 = \frac{(1+u)^2}{|1+u|^2} = \frac{(1+u)^2}{(1+u)(1+u)^*} = \frac{1+u}{1+u^*}.$$

Now since u is a direction, we know that $uu^* = 1$, and hence

$$\frac{1+u}{1+u^*} = \frac{uu^*+u}{1+u^*} = \frac{u(u^*+1)}{(1+u^*)} = u. \ \|$$

6.24 Corollary. *Every complex number has a square root.*

Proof: Let $\alpha \in \mathbf{C}$. If $\alpha = 0$, then clearly α has a square root. If $\alpha \neq 0$, let ru be the polar decomposition for α. If $u \neq -1$, then $\pm r^{\frac{1}{2}}\left(\dfrac{1+u}{|1+u|}\right)$ are square roots of α. If $u = -1$, then $\pm r^{\frac{1}{2}} i$ are square roots of α. $\|$

6.25 Example. We will find the square roots of $21-20i$. Let $\alpha = 21-20i$. Then

$$|\alpha| = \sqrt{21^2+20^2} = \sqrt{441+400} = \sqrt{841} = 29.$$

Hence the polar decomposition for α is

$$\alpha = 29\left(\frac{21-20i}{29}\right) = ru \text{ where } r = 29 \text{ and } u = \frac{21-20i}{29}.$$

The square roots of α are

$$\begin{aligned}\pm r^{\frac{1}{2}}\left(\frac{1+u}{|1+u|}\right) &= \pm\sqrt{29}\left(\frac{1+\frac{21}{29}-\frac{20i}{29}}{|1+\frac{21}{29}-\frac{20i}{29}|}\right)\\ &= \pm\sqrt{29}\left(\frac{50-20i}{|50-20i|}\right) = \pm\sqrt{29}\left(\frac{5-2i}{|5-2i|}\right).\end{aligned}$$

Now $|5-2i| = \sqrt{25+4} = \sqrt{29}$, so the square roots of α are $\pm(5-2i)$.

6.26 Exercise. Find the square roots of $12+5i$. Write your answers in the form $a+bi$, where a and b are real.

Let $a, b \in \mathbf{R}$. There is a formula for the square root of $a+bi$ that allows you to say

$$\text{the square roots of } 2+4i \text{ are } \pm\left(\sqrt{\sqrt{5}+1}+i\sqrt{\sqrt{5}-1}\right) \tag{6.27}$$

and

$$\text{the square roots of } 6-2i \text{ are } \pm\left(\sqrt{\sqrt{10}+3}-i\sqrt{\sqrt{10}-3}\right). \tag{6.28}$$

6.29 Exercise. Verify that assertions (6.27) and (6.28) are correct.

6.30 Entertainment. Find the square root formula, and prove that it is correct. (There are at least three ways to do this. Method c) is probably the easiest.)

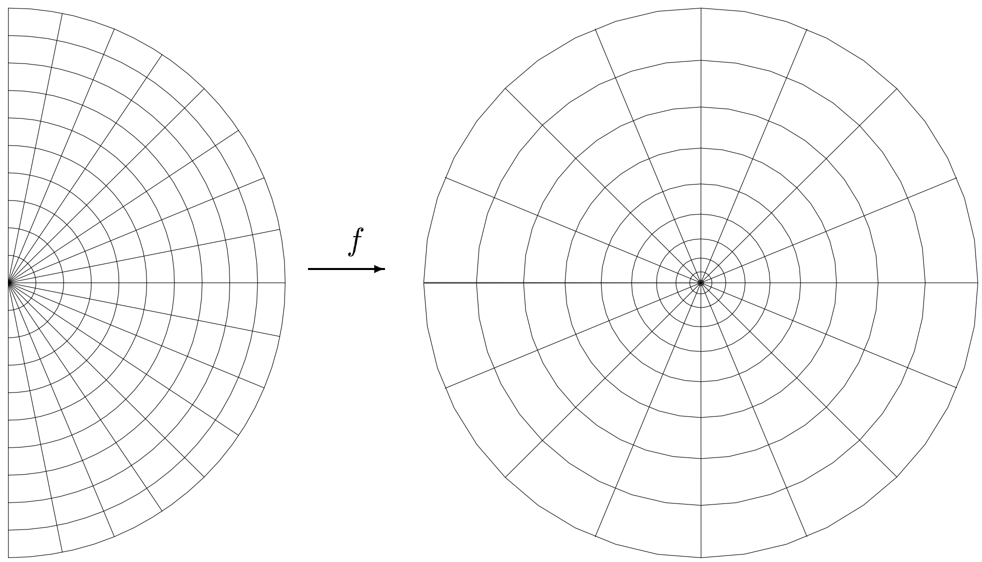
a) Suppose the square root is $c+di$, and equate the real and imaginary parts of $(c+di)^2$ and $a+bi$. Then solve for c and d and show that your solution works.

b) Let ru be the polar decomposition of $a+bi$. You know how to find a square root v for u, and $r^{\frac{1}{2}}v$ will be a square root of $a+bi$. Write this in the form $c+di$.

c) On the basis of (6.27) and (6.28), guess the formula, and show that it works.

6.5 Complex Functions

When one studies a function f from $\mathbf{R}$ to $\mathbf{R}$, one often gets information by looking at the graph of f, which is a subset of $\mathbf{R}\times\mathbf{R}$. If we consider a function $g\colon \mathbf{C}\to\mathbf{C}$, the graph of g is a subset of $\mathbf{C}\times\mathbf{C} = (\mathbf{R}\times\mathbf{R})\times(\mathbf{R}\times\mathbf{R})$, and $\mathbf{C}\times\mathbf{C}$ is a "4-dimensional" object which cannot be visualized. We will now discuss a method to represent functions from $\mathbf{C}$ to $\mathbf{C}$ geometrically.



Geometrical Representation of the Function $f(z) = z^2$.

6.31 Example ($f(z) = z^2$.) Let $f\colon \mathbf{C} \to \mathbf{C}$ be defined by $f(z) = z^2$. If z is a point in the circle $C(0, r)$, then $z = ru$ where u is a direction, and $f(z) = r^2u^2$ is a point in the circle $C(0, r^2)$ with radius r^2. Thus f maps circles of radius r about 0 into circles of radius r^2 about 0. Let u_0 be a direction in $\mathbf{C}$. If z is on the ray from 0 passing through u_0, then $z = ru_0$ for some $r \in \mathbf{R}^+$ so $f(z) = r^2u_0^2$, which is on the ray from 0 passing through u_0^2. Hence the ray making an angle θ with the positive real axis gets mapped by f to the ray making an angle 2θ with the positive x-axis.

The left part of the figure shows a network formed by semicircles of radius

$$r \in \{.1, .2, .3, \cdots, .9, 1\},$$

and rays making angles

$$\theta \in \left\{0, \pm\frac{\pi}{16}, \pm\frac{2\pi}{16}, \cdots, \pm\frac{8\pi}{16}\right\}$$

with the positive x-axis. The right part of the figure shows the network formed by circles of radius

$$r^2 \in \{.1^2, .2^2, \cdots, .9^2, 1\}$$

and rays making angles

$$2\theta \in \left\{0, \pm\frac{\pi}{8}, \pm\frac{2\pi}{8}, \cdots, \pm\frac{8\pi}{8}\right\}$$

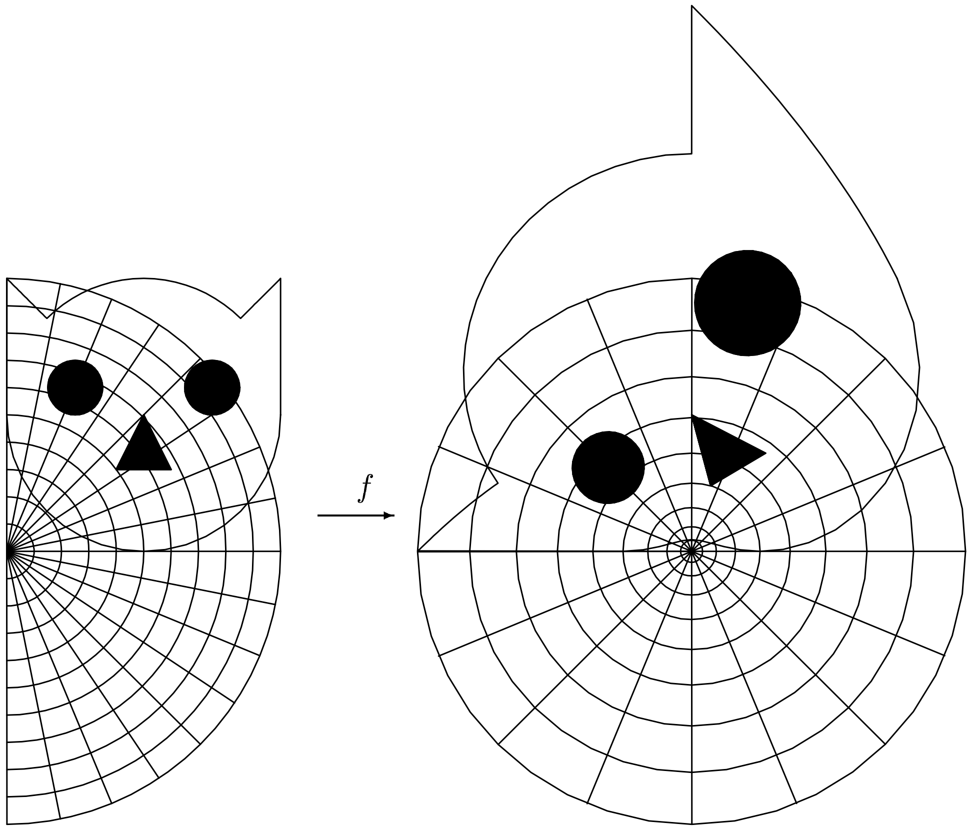
with the positive x-axis. f maps each semicircle in the left part of the figure to a circle in the right part, and f maps each ray in the left part to a ray in the right part. Also f maps each curvilinear rectangle on the left to a curvilinear rectangle on the right. Notice that $f(i) = f(-i)$, and in general $f(z) = f(-z)$, so if we know how f maps points in the right half plane, we know how it maps points in the left half plane. The function f maps the right half plane $\{x > 0\}$ onto $\mathbf{C} \setminus \Big((\text{ negative real axis }) \cup \{0\} \Big)$.

6.32 Definition (Image of a function.) Let S, T be sets, let $f \colon S \to T$, and let A be a subset of $\text{dom}(f)$. We define

$$f(A) = \{f(a) \colon a \in A\}$$

and we call $f(A)$ the *image of A under f*. We call $f\,(\text{dom}(f))$ the *image of f*.

6.33 Example ($f(z) = z^2$, continued) In the figure on page 118, the right half of the figure is the image of the left half under the function f. The figure on page 120, shows the image of a cat-shaped set under f. The cat on the left lies in the first quadrant, so its square lies in the first two quadrants. The tip of the right ear is $1 + i = \sqrt{2}\left(\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}\right)$, with length $\sqrt{2}$, and with direction making an angle $\dfrac{\pi}{4}$ with the positive real axis. The image of the right ear has length $(\sqrt{2})^2 = 2$ and makes an angle $\dfrac{\pi}{2}$ with the positive x-axis. You should examine how the parts of the cat in each curvilinear rectangle on the left part of the figure correspond to their images on the right part.



The Square of a Cat

6.34 Exercise. Let C be the cat shown in the left part of the above figure. Sketch the image of C under each of the functions g, h, k below:

a) $g(z) = 2z$.

b) $h(z) = iz$.

c) $k(z) = 2iz$.

6.35 Exercise. Let C be the cat shown in the left part of the above figure. Sketch the image of C under G, where $G(z) = -z^2$.

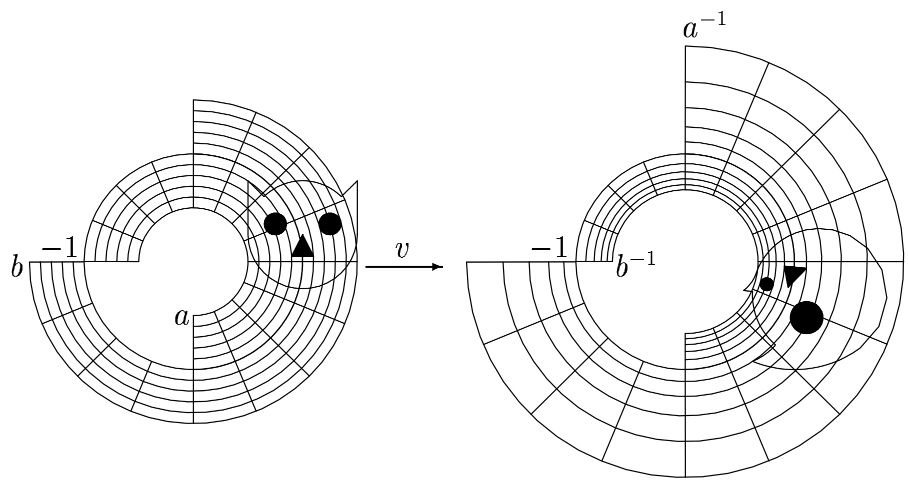
6.36 Exercise. Let z be a direction in $\mathbf{C}$; i.e., let $z \in C(0,1)$. Show that $z^* = z^{-1}$.

6.37 Example. Let $v(z) = \frac{1}{z}$ for all $z \in \mathbf{C}\backslash\{0\}$. If z is in the circle of radius r, then $z = ru$ for some direction u, and $|v(z)| = \left|\frac{1}{ru}\right| = \frac{1}{|r|\ |u|} = \frac{1}{|r|}$, so v takes points in the circle of radius r about 0 to points in the circle of radius $\frac{1}{r}$ about 0.

Let u_0 be a direction. If z is in the ray from 0 through u_0, then $z = ru_0$ for some $r \in \mathbf{R}^+$, so $v(z) = \frac{1}{r}u_0^{-1} = \frac{1}{r}u_0^*$. We noted earlier that u_0^* is the reflection of u_0 about the real axis, so v maps the ray making angle θ with the positive real axis into the ray making angle $-\theta$ with the positive real axis. Thus v maps the network of circles and lines in the left half of the figure into the network on the right half.

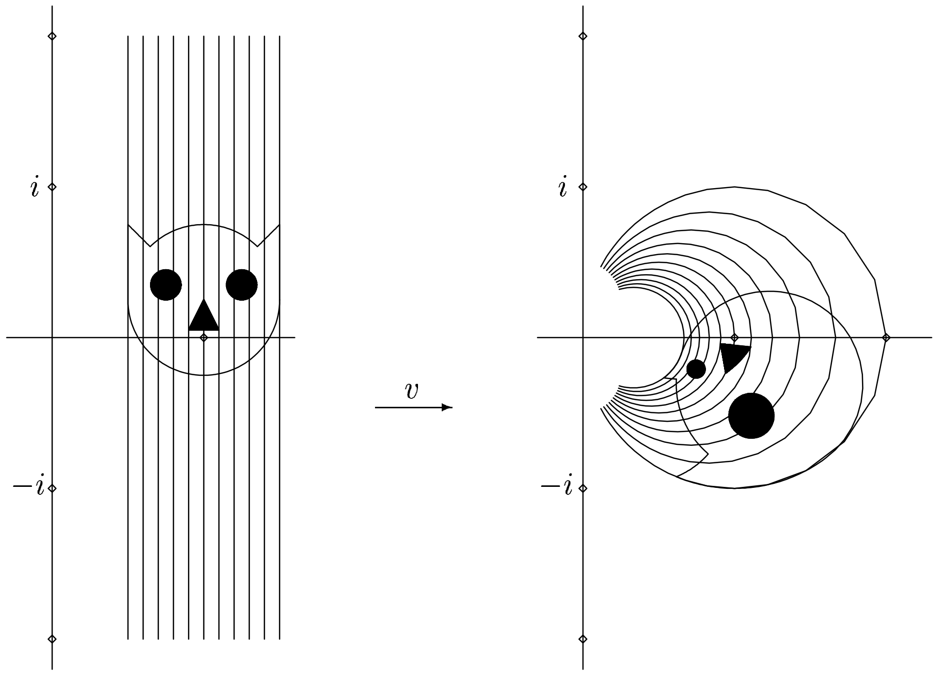
The circular arcs in the left half of the figure have radii

$$r \in \{.5, .6, .7, \cdots, 1.4, 1.5\}.$$



The Inverse of a Cat

Let's see how v maps the vertical line $x = a$ $(a \neq 0)$, $a \in \mathbf{R}$. We know that $v(a) = \frac{1}{a}$ and v maps points in the upper half plane to points in the lower half plane. Points far from the origin get mapped to points near to the origin. I claim that v maps the line $x = a$ into the circle with center $\frac{1}{2a}$ and radius $\frac{1}{2|a|}$.



v maps vertical lines to circles

Let $L_a = \{z\colon \mathrm{Re}(z) = a\} = \{a + iy\colon y \in \mathbf{R}\}$, so L_a is the set of points in the line $x = a$. Then

$$\begin{aligned} z \in L_a &\iff z = a + iy \text{ for some } y \in \mathbf{R} \\ &\implies \frac{1}{z} - \frac{1}{2a} = \frac{2a - z}{2az} = \frac{2a - (a + iy)}{2a(a + iy)} = \frac{a - iy}{2a(a + iy)} \\ &\implies \left|\frac{1}{z} - \frac{1}{2a}\right| = \left|\frac{1}{2a} \cdot \frac{a - iy}{a + iy}\right| = \frac{1}{|2a|}\frac{|a - iy|}{|a + iy|} = \frac{1}{|2a|}, \end{aligned}$$

since $|w| = |w^*|$ for all $w \in \mathbf{C}$. Hence,

$$z \in L_a \implies \left|\frac{1}{z} - \frac{1}{2a}\right| = \frac{1}{|2a|} \implies \frac{1}{z} \in C\left(\frac{1}{2a}, \frac{1}{|2a|}\right),$$

and v maps every point in L_a into $C\left(\frac{1}{2a}, \frac{1}{|2a|}\right)$. Now I claim that every point in $C\left(\frac{1}{2a}, \frac{1}{|2a|}\right)$ (except for 0) is equal to $v(z)$ for some $z \in L_a$.

Since $w = v\left(v(w)\right)$, it will be sufficient to show that if $w \in C\left(\frac{1}{2a}, \frac{1}{|2a|}\right)\backslash\{0\}$, then $v(w) \in L_a$. I want to show

$$\left(\left|w - \frac{1}{2a}\right| = \frac{1}{|2a|} \text{ and } w \neq 0\right) \implies \frac{1}{w} = a + iy \text{ for some } y \in \mathbf{R}.$$

Well, suppose $\left|w - \frac{1}{2a}\right| = \frac{1}{|2a|}$, and let $\frac{1}{w} = A + iB$ where $A, B \in \mathbf{R}$. Then $w = \frac{1}{A+iB} = \frac{A - iB}{A^2 + B^2}$, so

$$\begin{aligned}
\left|w - \frac{1}{2a}\right| = \frac{1}{|2a|} &\implies \left|w - \frac{1}{2a}\right|^2 = \frac{1}{4a^2} \\
&\implies \left|\frac{A - iB}{A^2 + B^2} - \frac{1}{2a}\right|^2 = \frac{1}{4a^2} \\
&\implies \left|\left(\frac{A}{A^2 + B^2} - \frac{1}{2a}\right) - \frac{iB}{A^2 + B^2}\right|^2 = \frac{1}{4a^2} \\
&\implies \frac{A^2}{\left(A^2 + B^2\right)^2} - \frac{A}{a(A^2 + B^2)} + \frac{1}{4a^2} + \frac{B^2}{(A^2 + B^2)^2} = \frac{1}{4a^2} \\
&\implies \frac{A^2 + B^2}{(A^2 + B^2)^2} = \frac{A}{a(A^2 + B^2)} \\
&\implies A = a,
\end{aligned}$$

so (by definition of A)

$$\left|w - \frac{1}{2a}\right| = \frac{1}{2a} \implies \frac{1}{w} = a + iB \text{ where } B \in \mathbf{R}. \parallel$$

6.38 Exercise. The argument above does not apply to the vertical line $x = 0$. Let $L_0 = \{iy\colon y \in \mathbf{R}\}$. Where does the reciprocal function v map $L_0\backslash\{0\}$?

6.39 Entertainment. Let $v(z) = \frac{1}{z}$ for all $z \in \mathbf{C}\backslash\{0\}$. Show that v maps horizontal lines $y = c$ $(c \neq 0)$ into circles that pass through the origin. Sketch the images of the lines

$$x = j, \text{ where } j \in \{0, \pm 1, \pm 2, \pm 3\}$$

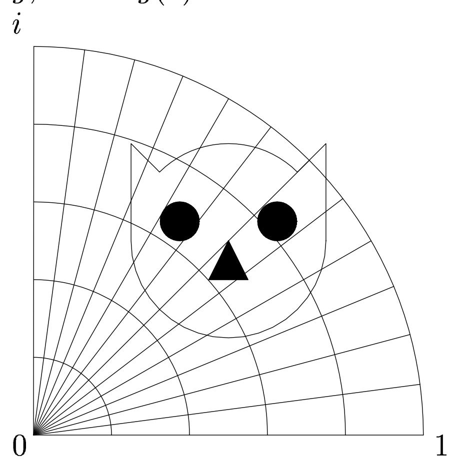
and the lines

$$y = j, \text{ where } j \in \{0, \pm1, \pm2, \pm3\}$$

on one set of axes using a compass. If you've done this correctly, the circles should intersect at right angles.

6.40 Exercise.

a) Sketch the image of the network of lines and circular arcs shown below under the function g, where $g(z) = z^3$ for all $z \in \mathbf{C}$.



b) Cube the cat in the picture.

6.41 Note. De Moivre's formula $(\cos(\theta) + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, was first stated in this form by Euler in 1749 ([46, pp. 452-454]). Euler named the formula after Abraham De Moivre (1667-1754) who never explicitly stated the formula, but used its consequences several times ([46, pp. 440-450]).

The method for finding mth roots of complex numbers:

$$[r(\cos\theta + i\sin\theta)]^{\frac{1}{m}} = r^{\frac{1}{m}}\left[\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\right]$$

was introduced by Euler in 1749 [46, pp.452-454].

The idea of illustrating functions from the plane to the plane by distorting cat faces is due to Vladimir Arnold (1937-??), and the figures are sometimes called "Arnold Cats". Usually Arnold cats have black faces and white eyes and noses, as in [3, pp.6-9].