Chapter 16

Fundamental Theorem of Calculus

16.1 Definition (Nice functions.) I will say that a real valued function f defined on an interval $[a,b]$ is a *nice function* on $[a,b]$, if f is continuous on $[a,b]$ and integrable on every subinterval of $[a,b]$.

Remark: We know that piecewise monotonic continuous functions on $[a,b]$ are nice. It turns out that every continuous function on $[a,b]$ is nice, but we are not going to prove this. The theorems stated in this chapter for nice functions are usually stated for continuous functions. You can find a proof that every continuous function on an interval $[a,b]$ is integrable on $[a,b]$ (and hence that every continuous function on $[a,b]$ is nice on $[a,b]$) in [44, page 246] or in [1, page 153]. However both of these sources use a slightly different definition of continuity and of integral than we do, so you will need to do some work to translate the proofs in these references into proofs in our terms. You might try to prove the result yourself, but the proof is rather tricky. For all the applications we will make in this course, the functions examined will be continuous and piecewise monotonic so the theorems as we prove them are good enough.

16.2 Exercise. Can you give an example of a continuous function on a closed interval that is *not* piecewise monotonic? You may describe your example rather loosely, and you do not need to prove that it is continuous.

16.3 Theorem (Fundamental theorem of calculus I.) *Let g be a nice function on $[a,b]$. Suppose G is an antiderivative for g on $[a,b]$. Then G is* *an indefinite integral for g on $[a,b]$; i.e.,*

$$\int_p^q g = G(q) - G(p) \text{ for all } p, q \in [a,b]. \tag{16.4}$$

Proof: By the definition of antiderivative, G is continuous on $[a,b]$ and $G' = g$ on (a,b). Let p, q be arbitrary points in $[a,b]$. I will suppose $p < q$. (Note that if (16.4) holds when $p < q$, then it holds when $q < p$, since both sides of the equation change sign when p and q are interchanged. Also note that the theorem clearly holds for $p = q$.)

Let $P = \{x_0, \cdots, x_m\}$ be any partition of $[p,q]$, and let i be an integer with $1 \leq i \leq m$. If $x_{i-1} < x_i$ we can apply the mean value theorem to G on $[x_{i-1}, x_i]$ to find a number $s_i \in (x_{i-1}, x_i)$ such that

$$g(s_i)(x_i - x_{i-1}) = G'(s_i)(x_i - x_{i-1}) = G(x_i) - G(x_{i-1}).$$

If $x_i = x_{i-1}$, let $s_i = x_i$. Then $S = \{s_i, \cdots, s_m\}$ is a sample for P such that

$$\begin{aligned} \sum(g, P, S) &= \sum_{i=1}^{m} g(s_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{m} G(x_i) - G(x_{i-1}) \\ &= G(x_m) - G(x_0) = G(q) - G(p). \end{aligned}$$

We have shown that for every partition P of $[p,q]$ there is a sample S for P such that

$$\sum(g, P, S) = G(q) - G(p).$$

Let $\{P_n\}$ be a sequence of partitions for $[p,q]$ such that $\{\mu(P_n)\} \to 0$, and for each $n \in \mathbf{Z}^+$ let S_n be a sample for P_n such that

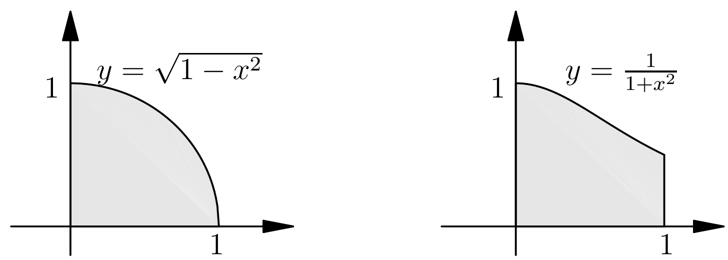
$$\sum(g, P_n, S_n) = G(q) - G(p).$$

Then, since g is integrable on $[q,p]$,

$$\begin{aligned} \int_p^q g &= \lim\{\sum(g, P_n, S_n)\} \\ &= \lim\{G(q) - G(p)\} = G(q) - G(p). \parallel \end{aligned}$$

16.5 Example. The fundamental theorem will allow us to evaluate many integrals easily. For example, we know that $\dfrac{d}{dx}\Big(\arctan(x)\Big) = \dfrac{1}{1+x^2}$. Hence, by the fundamental theorem,

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2}dx &= \arctan(x)\ |_0^1 = \arctan(1) - \arctan(0)\\ &= \frac{\pi}{4} - 0 = \frac{\pi}{4}.\end{aligned}$$



Two sets with the same area

This says that the two sets

$$\{(x,y)\colon 0 \le x \le 1 \text{ and } 0 \le y \le \sqrt{1-x^2}\}$$

and

$$\{(x,y)\colon 0 \le x \le 1 \text{ and } 0 \le y \le \frac{1}{1+x^2}\}$$

have the same area – a rather remarkable result.

16.6 Theorem (Mean value theorem for integrals.) *Let f be a nice function on an interval $[p,q]$, where $p < q$. Then there is a number $c \in (p,q)$ such that*

$$\int_p^q f = f(c)(q-p) \ \textit{i.e.,}\ f(c) = \frac{1}{q-p}\int_p^q f.$$

Proof: Since f is continuous on $[p,q]$ we can find numbers $r, s \in [p,q]$ such that

$$f(r) \le f(x) \le f(s) \text{ for all } x \in [p,q].$$

By the inequality theorem for integrals

$$\int_p^q f(r) \leq \int_p^q f \leq \int_p^q f(s);$$

(here $f(r)$ and $f(s)$ denote constant functions) i.e.,

$$f(r)(q-p) \leq \int_p^q f \leq f(s)(q-p),$$

i.e.,

$$f(r) \leq \frac{1}{q-p}\int_p^q f \leq f(s).$$

We can now apply the intermediate value property to f on the interval whose endpoints are r and s to get a number c between r and s such that

$$f(c) = \frac{1}{q-p}\int_p^q f.$$

The number c is in the interval (p, q), so we are done. $\|$

16.7 Corollary. *Let f be a nice function on a closed interval whose endpoints are p and q where $p \neq q$. Then there is a number c between p and q such that*

$$f(c) = \frac{1}{q-p}\int_p^q f.$$

16.8 Exercise. Explain how corollary 16.7 follows from theorem 16.6. (There is nothing to show unless $q < p$)

16.9 Lemma. *Let f be a function such that f is integrable on every subinterval of $[a, b]$. Let $c \in [a, b]$ and let*

$$F(x) = \int_c^x f \text{ for all } x \in [a, b].$$

Then F is continuous on $[a, b]$.

Proof: Let $t \in [a, b]$. I will show that F is continuous at t. Since f is integrable on $[a, b]$ there is a number M such that

$$-M \leq f(x) \leq M \text{ for all } x \in [a, b].$$

By the corollary to the inequality theorem for integrals (8.17), it follows that

$$\Big|\int_s^t f\Big| \leq M|s-t|$$

for all $s,t \in [a,b]$. Thus, for all $s,t \in [a,b]$,

$$0 \leq |F(s)-F(t)| = \Big|\int_c^s f - \int_c^t f\Big| = \Big|\int_t^s f\Big| \leq M|s-t|.$$

Now $\lim\limits_{s\to t} M|s-t| = 0$, so by the squeezing rule for limits of functions, $\lim\limits_{s\to t} |F(s)-F(t)| = 0$. It follows that $\lim_{s\to t} F(s) = F(t)$. |||.

16.10 Theorem (Fundamental theorem of calculus II.) *Let f be a nice function on $[a,b]$, and let $c \in [a,b]$. Let*

$$G(x) = \int_c^x f \textit{ for all } x \in [a,b].$$

Then G is an antiderivative for f, i.e.

$$\frac{d}{dx}\int_c^x f = \frac{d}{dx}\int_c^x f(t)\,dt = f(x). \tag{16.11}$$

In particular, every nice function on $[a,b]$ has an antiderivative on $[a,b]$.

Proof: Let

$$G(x) = \int_c^x f \text{ for all } x \in [a,b]$$

and let t be a point in (a,b). Let $\{x_n\}$ be any sequence in $[a,b]\setminus\{t\}$ such that $\{x_n\} \to t$. Then

$$\begin{aligned}\frac{G(x_n)-G(t)}{x_n-t} &= \frac{1}{x_n-t}\Big[\int_c^{x_n} f - \int_c^t f\Big]\\ &= \frac{1}{x_n-t}\int_t^{x_n} f.\end{aligned}$$

By the mean value theorem for integrals, there is a number s_n between x_n and t such that

$$\frac{G(x_n)-G(t)}{x_n-t} = f(s_n).$$

Now

$$0 \leq |s_n - t| \leq |x_n - t| \text{ for all } n$$

and since $\{|x_n - t|\} \to 0$, we have $\{|s_n - t|\} \to 0$, by the squeezing rule for sequences. Since f is continuous, we conclude that $\{f(s_n)\} \to f(t)$; i.e.,

$$\Big\{\frac{G(x_n) - G(t)}{x_n - t}\Big\} \to f(t);$$

i.e.,

$$\lim_{x \to t} \frac{G(x) - G(t)}{x - t} = f(t).$$

This proves that $G'(t) = f(t)$ for $t \in (a, b)$. In addition G is continuous on $[a, b]$ by lemma 16.9. Hence G is an antiderivative for f on $[a, b]$. |||

Remark Leibnitz's statement of *the fundamental principle of the calculus* was the following:

> Differences and sums are the inverses of one another, that is to say, the sum of the differences of a series is a term of the series, and the difference of the sums of a series is a term of the series; and I enunciate the former thus, $\int dx = x$, and the latter thus, $d \int x = x$[34, page 142].

To see the relation between Leibnitz's formulas and ours, in the equation $d \int x = x$, write $x = ydt$ to get $d \int ydt = ydt$, or $\dfrac{d}{dt} \displaystyle\int ydt = y$. This corresponds to equation (16.11). Equation (16.4) can be written as

$$\int_p^q \frac{dG}{dx} dx = G(q) - G(p).$$

If we cancel the dx's (in the next chapter we will show that this is actually justified!) we get $\int_p^q dG = G(q) - G(p)$. This is not quite the same as $\int dx = x$. However if you choose the origin of coordinates to be $(p, G(p))$, then the two formulas coincide.

To emphasize the inverse-like relation between differentiation and integration, I will restate our formulas for both parts of the the fundamental theorem, ignoring all hypotheses:

$$\frac{d}{dx} \int_c^x f(t)dt = f(x) \text{ and } \int_c^x \frac{d}{dt} f(t)dt = f(x) - f(c).$$

By exploiting the ambiguous notation for indefinite integrals, we can get a form almost identical with Leibniz's:

$$\frac{d}{dx}\int f(x)dx = f(x) \text{ and } \int \frac{d}{dx}f(x)dx = f(x).$$

16.12 Example. Let

$$\begin{aligned} F(x) &= \int_1^x e^{t^2}\,dt, \\ G(x) &= \int_1^{x^3} e^{t^2}\,dt, \\ H(x) &= e^{-x^2}\int_1^x e^{t^2}\,dt. \end{aligned}$$

We will calculate the derivatives of F, G, and H. By the fundamental theorem,

$$F'(x) = e^{x^2}.$$

Now $G(x) = F(x^3)$, so by the chain rule,

$$\begin{aligned} G'(x) &= F'(x^3)\cdot 3x^2 \\ &= e^{(x^3)^2}\cdot 3x^2 = 3x^2 e^{x^6}. \end{aligned}$$

We have $H(x) = e^{-x^2}F(x)$, so by the product rule,

$$\begin{aligned} H'(x) &= e^{-x^2}F'(x) + e^{-x^2}(-2x)F(x) \\ &= e^{-x^2}e^{x^2} + e^{-x^2}(-2x)\int_0^x e^{t^2}\,dt \\ &= 1 - 2xe^{-x^2}\int_0^x e^{t^2}\,dt. \end{aligned}$$

16.13 Exercise. Calculate the derivatives of the following functions. Simplify your answers as much as you can.

a) $F(x) = \displaystyle\int_0^x \frac{1}{\sqrt{1+t^2}}\,dt$

b) $G(x) = \displaystyle\int_0^x \frac{1}{\sqrt{t^2-1}}\,dt$

c) $H(x) = \displaystyle\int_2^x \frac{1}{\sqrt{t^2-1}}\,dt$

d) $K(x) = \displaystyle\int_1^{\sinh(x)} \frac{1}{\sqrt{1+t^2}}\, dt$

e) $L(x) = \displaystyle\int_2^{\cosh(x)} \frac{1}{\sqrt{t^2-1}}\, dt.$

(We defined cosh and sinh in exercise 14.56.) Find simple formulas (not involving any integrals) for K and for L.

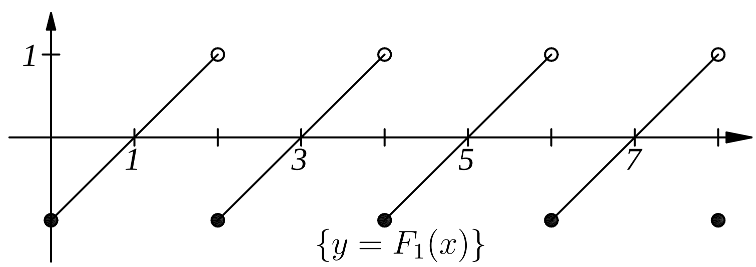
16.14 Exercise. Use the fundamental theorem of calculus to find

a) $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\, dx.$

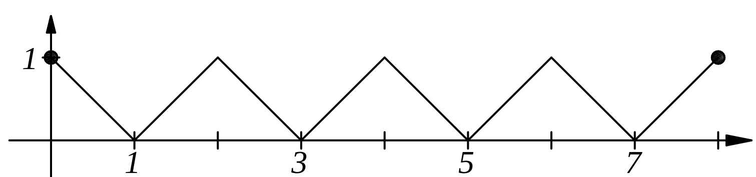
b) $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}\, dx.$

c) $\displaystyle\int_0^{1} e^x\, dx.$

16.15 Exercise. Let F_1 and F_2 be the functions whose graphs are shown below:



$\{y = F_1(x)\}$



$\{y = F_2(x)\}$

Let $G_i(x) = \displaystyle\int_0^x F_i(t)\, dt$ for $0 \le t \le 8$. Sketch the graphs of G_1 and G_2. Include some discussion about why your answer is correct.