Chapter 15

The Second Derivative

15.1 Higher Order Derivatives

15.1 Definition (Higher order derivatives.) Let f be a function whose domain is a subset of $\mathbf{R}$. We define a function f' (called the *derivative of f*) by

$$\text{domain}(f') = \{x \in \text{dom}(f) : f'(x) \text{ exists}\}.$$

and for all $x \in \text{dom}(f)$, the value of f' at x is the derivative $f'(x)$. We may also write $f^{(1)}$ for f'. Since f' is itself a function, we can calculate its derivative: this derivative is denoted by f'' or $f^{(2)}$, and is called the *second derivative of f*. For integers $n \geq 2$ we define

$$f^{(n+1)} = (f^{(n)})'. \tag{15.2}$$

and we call $f^{(n)}$ the *nth derivative of f*. We also define

$$f^{(0)} = f.$$

In Leibniz's notation we write

$$\frac{d^n f}{dx^n} = f^{(n)}, \text{ or } \frac{d^n}{dx^n} f = f^{(n)}, \text{ or } \left(\frac{d}{dx}\right)^{(n)} f(x) = f^{(n)}(x) \text{ or } \frac{d^n f}{dx^n} = f^{(n)}(x),$$

so that equation (15.2) becomes

$$\frac{d^{n+1} f}{dx^{n+1}} = \frac{d}{dx}\left(\frac{d^n f}{dx^n}\right).$$

If a and b are real numbers, and f and g are functions then from known properties of the derivative we can show that

$$(af+bg)^{(n)} = af^{(n)} + bg^{(n)} \text{ on } \mathrm{dom}(f^{(n)}) \cap \mathrm{dom}(g^{(n)}).$$

or

$$\frac{d^n}{dx^n}(af+bg) = a\frac{d^n f}{dx^n} + b\frac{d^n f}{dx^n}.$$

15.3 Examples. If $h(x) = \sin(\omega x)$, where $\omega \in \mathbf{R}$, then

$$\begin{aligned} h'(x) &= \omega\cos(\omega x), \\ h''(x) &= -\omega^2\sin(\omega x), \\ h^{(3)}(x) &= -\omega^3\cos(\omega x), \\ h^{(4)}(x) &= \omega^4\sin(\omega x) = \omega^4 h(x). \end{aligned}$$

It should now be apparent that

$$h^{(4n+k)}(x) = \omega^{4n}h^{(k)}(x) \text{ for } k = 0,1,2,3.$$

so that

$$h^{(98)}(x) = h^{(4\cdot 24+2)}(x) = \omega^{96}h^{(2)}(x) = -\omega^{98}\sin(\omega x).$$

If

$$g(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

then

$$\begin{aligned} g'(x) &= 1 + x + \frac{x^2}{2!}, \\ g''(x) &= 1 + x, \\ g^{(3)}(x) &= 1, \\ g^{(n)}(x) &= 0 \text{ for } n \in \mathbf{Z}_{\geq 4}. \end{aligned}$$

If $y = \ln(x)$ then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{x}, \\
\frac{d^2y}{dx^2} &= -\frac{1}{x^2}, \\
\frac{d^3y}{dx^3} &= \frac{2}{x^3}.
\end{aligned}$$

15.4 Exercise. Calculate $g^{(5)}(t)$ if $g(t) = t^4 \ln(t)$.

15.5 Exercise. Let $g(t) = tf(t)$. Calculate $g'(t)$, $g''(t)$, $g^{(3)}(t)$ and $g^{(4)}(t)$ in terms of $f(t)$, $f'(t)$, $f''(t)$, $f^{(3)}(t)$ and $f^{(4)}(t)$. What do you think is the formula for $g^{(n)}(t)$?

15.6 Exercise. Find $\dfrac{d^2y}{dx^2}$ if $y = 1/(x^2 - 1)$.

15.7 Exercise. Find $f''(x)$ if $f(x) = e^{\frac{1}{x^2}} = \exp\left(\dfrac{1}{x^2}\right)$.

15.8 Exercise. Suppose $f''(x) = 0$ for all $x \in \mathbf{R}$. What can you say about f?

15.9 Exercise. Let f and g be functions such that $f^{(2)}$ and $g^{(2)}$ are defined on all of $\mathbf{R}$. Show that

$$(fg)^{(2)} = fg^{(2)} + 2f^{(1)}g^{(1)} + f^{(2)}g.$$

Find a similar function for $(fg)^{(3)}$ (assuming that $f^{(3)}$ and $g^{(3)}$ are defined.)

In Leibniz's calculus, d^2f or ddf was actually an infinitely small quantity that was so much smaller than dx that the quotient $\dfrac{d^2f}{dx}$ was zero, and $\dfrac{d^2f}{dx^2}$ was obtained by multiplying dx by itself and then dividing the result into d^2f. Leibniz also used notations like $\dfrac{ddy}{ddx}$ and $\dfrac{dxds}{ddy}$ for which our modern notation has no counterparts. Leibniz considered the problem of defining a meaning for $d^{\frac{1}{2}}f$, but he did not make much progress on this problem. Today there is considerable literature on fractional derivatives. A brief history of the subject can be found in [36, ch I and ch VIII].

15.10 Exercise. Let a be a real number. Show that for $k = 0, 1, 2, 3$

$$\frac{d^k}{dx^k} e^{ax} = a^k e^{ax}. \tag{15.11}$$

After doing this it should be clear that equation (15.11), in fact holds for all $n \in \mathbf{Z}_{\geq 0}$ (this can be proved by induction). Now suppose that $a > 0$ and we will *define*

$$\frac{d^r}{dx^r} e^{ax} = a^r e^{ax} \text{ for all } r \in \mathbf{R}. \tag{15.12}$$

Show that then for all p and q in $\mathbf{R}$,

$$\left(\frac{d}{dx}\right)^p \left(\left(\frac{d}{dx}\right)^q (e^{ax})\right) = \left(\frac{d}{dx}\right)^{p+q} (e^{ax}).$$

Find $\left(\frac{d}{dx}\right)^{\frac{1}{2}} e^{3x}$ and $\left(\frac{d}{dx}\right)^{\frac{1}{2}} e^{5x}$. What do you think $\left(\frac{d}{dx}\right)^{\frac{1}{2}} \left(3e^{3x} + 4e^{5x}\right)$ should be?

Equation (15.12) was the starting point from which Joseph Liouville (1809–1882) developed a theory of fractional calculus[36, pp 4-6].

15.13 Exercise. Let a and b be real numbers. Show that for $k = 0, 1, 2, 3$

$$\left(\frac{d}{dx}\right)^k \cos(ax + b) = a^k \cos(ax + b + \frac{k\pi}{2}). \tag{15.14}$$

After doing this exercise it should be clear that in fact equation (15.14) holds for all $k \in \mathbf{Z}_{\geq 0}$ (this can be proved by induction). Now suppose that $a > 0$, and we will *define*

$$\left(\frac{d}{dx}\right)^r \cos(ax + b) = a^r \cos(ax + b + \frac{r\pi}{2}) \text{ for all } r \in \mathbf{R}. \tag{15.15}$$

Show that for all p and q in $\mathbf{R}$

$$\left(\frac{d}{dx}\right)^p \left(\left(\frac{d}{dx}\right)^q \cos(ax + b)\right) = \left(\frac{d}{dx}\right)^{p+q} \cos(ax + b).$$

Equation (15.15) was used as the starting point for a definition of fractional derivatives for general functions, by Joseph Fourier (1768–1830)[36, page 3].

15.2 Acceleration

15.16 Definition (Acceleration.) If a particle **p** moves in a straight line so that its position at time t is $h(t)$, we have defined its velocity at time t to be $h'(t)$. We now define its *acceleration* at time t to be $h''(t)$, so that acceleration is the derivative of velocity. Thus if a particle moves with a constant acceleration of $1 \frac{\text{ft./sec.}}{\text{sec.}}$, then every second its velocity increases by one ft./sec.

15.17 Example. A mass on the end of a spring moves so that its height at time t is $-A\cos(\omega t)$, where A and ω are positive numbers. If we denote its velocity at time t by $v(t)$, and its acceleration at time t by $a(t)$ then

$$\begin{aligned} h(t) &= -A\cos(\omega t) \\ v(t) = h'(t) &= A\omega\sin(\omega t) \\ a(t) = v'(t) &= A\omega^2\cos(\omega t) \end{aligned}$$

From this we see that the acceleration is always of opposite sign from the position: when the mass is above the zero position it is being accelerated downward, and when it is below its equilibrium position it is being accelerated upward. Also we see that the magnitude of the acceleration is largest when the velocity is 0.

15.18 Definition (Acceleration due to gravity.) If a particle **p** moves near the surface of the earth, acted on by no forces except the force due to gravity, then **p** will move with a constant acceleration $-g$ which is independent of the mass of **p**. The value of g is

$$g = \frac{32\text{ft./sec.}}{\text{sec.}}(\text{approx.}) \quad \text{or} \quad g = \frac{9.8\text{meter/sec.}}{\text{sec.}}(\text{approx.}).$$

We call g *the acceleration due to gravity*. Actually, the value of g varies slightly over the surface of the earth, so there is no exact value for g. The law just described applies in situations when air resistance and buoyancy can be neglected. It describes the motion of a falling rock well, but it does not describe a falling balloon.

Remark: When I solve applied problems, I will usually omit all units (e.g. feet or seconds) in my calculations, and will put them in only in the final answers.

15.19 Example. A juggler J tosses a ball vertically upward from a height of 4 feet above the ground with a speed of 16 ft./sec. Let $h(t)$ denote the height of the ball above the ground at time t. We will set our clock so that $t = 0$ corresponds to the time of the toss:

$$h(0) = 4; \qquad h'(0) = 16.$$

We will suppose that while the ball is in the air, its motion is described by a differentiable function of t. We assume that

$$h''(t) = -g = -32.$$

We know one function whose derivative is $-g$:

$$\text{if } s(t) = -gt, \text{ then } s'(t) = -g.$$

By the antiderivative theorem it follows that there is a constant v_0 such that

$$h'(t) = s(t) + v_0 = -gt + v_0.$$

Moreover we can calculate v_0 as follows:

$$(16 = h'(0) = -g \cdot 0 + v_0) \Longrightarrow (v_0 = 16).$$

Thus

$$h'(t) = -gt + 16.$$

We know a function whose derivative is $-gt + 16$:

$$\text{if } w(t) = -\frac{gt^2}{2} + 16t, \text{ then } w'(t) = -gt + 16.$$

Thus there is a constant h_0 such that

$$h(t) = w(t) + h_0 = -\frac{gt^2}{2} + 16t + h_0.$$

To find h_0 we set $t = 0$:

$$(4 = h(0) = -\frac{g \cdot 0^2}{2} + 16 \cdot 0 + h_0) \Longrightarrow (h_0 = 4).$$

Thus

$$h(t) = -\frac{gt^2}{2} + 16t + 4.$$

The ball will reach its maximum height when $h'(t) = 0$, i.e. when

$$t = \frac{16}{g} = \frac{16}{32} = \frac{1}{2}.$$

The maximum height reached by the ball is

$$h(\frac{1}{2}) = -\frac{1}{2} \cdot 32 \cdot (\frac{1}{2})^2 + \frac{16}{2} + 4 = 8,$$

so the ball rises to a maximum height of 8 feet above the ground.

15.20 Example (Conservation of energy.) Suppose that a particle $\mathbf{p}$ moves near the surface of the earth acted upon by no forces except the force of gravity. Let $v(t)$ and $h(t)$ denote respectively its height above the earth and its velocity at time t. Then

$$\frac{dv}{dt} = h''(t) = -g,$$

so

$$v\frac{dv}{dt} = -gv = -g\frac{dh}{dt}.$$

Now

$$v\frac{dv}{dt} = \frac{d}{dt}(\frac{1}{2}v^2),$$

so we have

$$\frac{d}{dt}(\frac{1}{2}v^2) = \frac{d}{dt}(-gh).$$

It follows that there is a constant K such that

$$\frac{1}{2}v^2 = -gh + K,$$

or

$$\frac{1}{2}v^2 + gh = K.$$

If m is the mass of the particle $\mathbf{p}$ then

$$\frac{1}{2}mv^2 + mgh = Km. \tag{15.21}$$

The quantity $\frac{1}{2}mv^2$ is called the *kinetic energy of* $\mathbf{p}$, and the quantity mgh is called the *potential energy* of $\mathbf{p}$. Equation (15.21) states that as $\mathbf{p}$ moves, the sum of its potential energy end its kinetic energy remains constant.

15.22 Exercise. A particle moves in a vertical line near the surface of the earth, acted upon by no forces except the force of gravity. At time 0 it is at height h_0, and has velocity v_0. Derive the formula for the height of the particle at time $t > 0$.

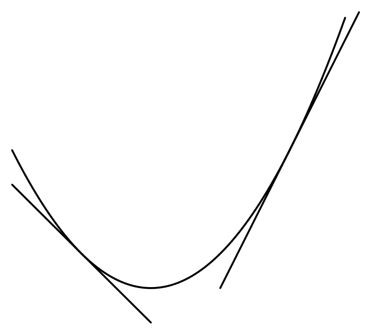
15.23 Exercise. The acceleration due to gravity on the moon is approximately

$$g_m = .17g$$

where g denotes the acceleration due to gravity on the earth. A juggler J on the moon wants to toss a ball vertically upward so that it rises 4 feet above its starting height. With what velocity should the ball leave J's hand?

15.3 Convexity

15.24 Definition (Convexity) Let f be a differentiable function on an interval (a, b). We say that f is *convex upward over* (a, b) or that f *holds water over* (a, b) if and only if for each point t in (a, b), the tangent line to graph(f) at $(t, f(t))$ lies below the graph of f.



convex upward curve (holds water)

Since the equation of the tangent line to graph(f) at $(t, f(t))$ is

$$y = f(t) + f'(t)(x - t),$$

the condition for f to be convex upward over (a, b) is that for all x and t in (a, b)

$$f(t) + f'(t)(x - t) \le f(x). \tag{15.25}$$

Condition (15.25) is equivalent to the two conditions:

$$f'(t) \leq \frac{f(x)-f(t)}{x-t} \text{ whenever } t < x,$$

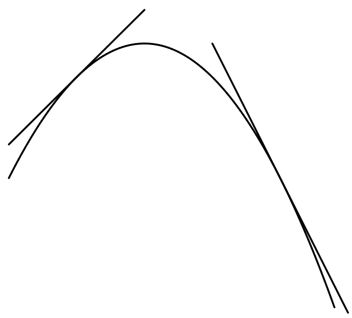
and

$$\frac{f(t)-f(x)}{t-x} \leq f'(t) \text{ whenever } x < t.$$

These last two conditions can be written as the single condition

$$f'(p) \leq \frac{f(q)-f(p)}{q-p} \leq f'(q) \text{ whenever } p < q. \tag{15.26}$$

We say that f is *convex downward over* (a, b), or that f *spills water over* (a, b) if and only if for each point t in (a, b), the tangent line to graph(f) at $(t, f(t))$ lies above the graph of f.



convex downward curve (spills water)

This condition is equivalent to the condition that for all points $p, q \in (a, b)$

$$f'(p) \geq \frac{f(q)-f(p)}{q-p} \geq f'(q) \text{ whenever } p < q.$$

15.27 Theorem. *Let f be a differentiable function over the interval (a, b). Then f is convex upward over (a, b) if and only if f' is increasing over (a, b). (and similarly f is convex downward over (a, b) if and only if f' is decreasing over (a, b).)*

Proof: If f is convex upward over (a, b), then it follows from (15.26) that f' is increasing over (a, b).

Now suppose that f' is increasing over (a, b). Let p, q be distinct points in (a, b). By the mean value theorem there is a point c between p and q such that

$$f'(c) = \frac{f(p) - f(q)}{p - q}.$$

If $p < q$ then $p < c < q$ so since f' is increasing over (a, b)

$$f'(p) \leq f'(c) \leq f'(q),$$

i.e.

$$f'(p) \leq \frac{f(p) - f(q)}{p - q} \leq f'(q).$$

Thus condition (15.26) is satisfied, and f is convex upward over (a, b).

15.28 Corollary. *Let f be a function such that $f''(x)$ exists for all x in the interval (a, b). If $f''(x) \geq 0$ for all $x \in (a, b)$ then f is convex upward over (a, b). If $f''(x) \leq 0$ for all $x \in (a, b)$ then f is convex downward over (a, b).*

15.29 Exercise. Prove one of the two statements in corollary 15.28.

15.30 Lemma (Converse of corollary 12.26) *Let f be a real function such that f is continuous on $[a, b]$ and differentiable on (a, b). If f is increasing on $[a, b]$, then $f'(x) \geq 0$ for all $x \in (a, b)$.*

Proof: let $p \in (a, b)$. Choose $\delta > 0$ such that $(p - \delta, p + \delta) \subset (a, b)$. Then $\{p + \frac{\delta}{2n}\}$ is a sequence such that

$$\{p + \frac{\delta}{2n}\} \to p,$$

and hence

$$\{\frac{f(p + \frac{\delta}{2n}) - f(p)}{(p + \frac{\delta}{2n}) - p}\} \to f'(p).$$

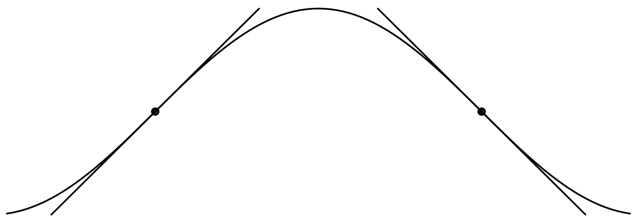
Since f is increasing on (a, b), we have

$$\frac{f(p + \frac{\delta}{2n}) - f(p)}{(p + \frac{\delta}{2n}) - p} \geq 0$$

for all $n \in \mathbf{Z}^+$, and it follows that

$$f'(p) \geq 0 \text{ for all } p \in (a, b). \;\|$$

15.31 Definition (Inflection point) Let f be a real function, and let $a \in \text{dom} f$. We say that a is a *point of inflection* for f if there is some $\epsilon > 0$ such that $(a-\epsilon, a+\epsilon) \subset \text{dom} f$, and f is convex upward on one of the intervals $(a-\epsilon, a)$, $(a, a+\epsilon)$, and is convex downward on the other.



inflection points

15.32 Theorem (Second derivative test for inflection points) *Let f be a real function, and let a be a point of inflection for f. If f'' is defined and continuous in some interval $(a-\delta, a+\delta)$ then $f''(a) = 0$.*

Proof: We will suppose that f is convex upward on the interval $(a-\delta, a)$ and is convex downward on $(a, a+\delta)$. (The proof in the case where these conditions are reversed is essentially the same). Then f' is increasing on $(a-\delta, a)$, and f' is decreasing on $(a, a+\delta)$. By (15.30), $f''(x) \geq 0$ for all $x \in (a-\delta, a)$, and $f''(x) \leq 0$ for all $x \in (a, a+\delta)$. We have

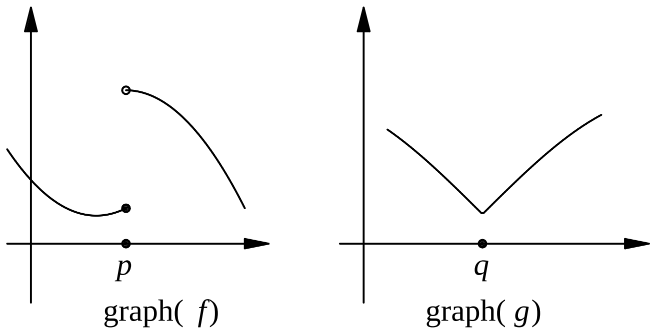
$$f''(a) = \lim\{f''(a + \frac{\delta}{2n})\} \leq 0,$$

and

$$f''(a) = \lim\{f''(a - \frac{\delta}{2n})\} \geq 0.$$

It follows that $f''(a) = 0$. |||

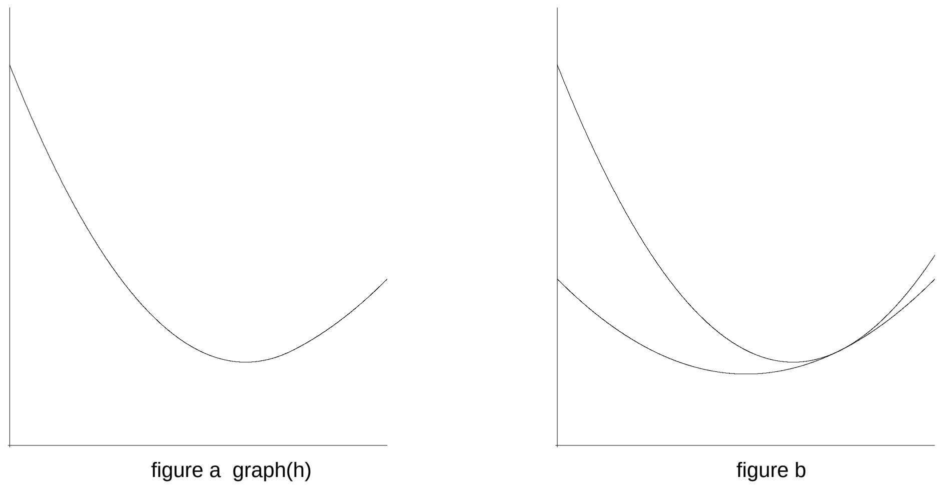
15.33 Example. When you look at the graph of a function, you can usually "see" the points where the second derivative changes sign. However, most people cannot "see" points where the second derivative is undefined.



By inspecting graph(f), you can see that f has a discontinuity at p.

By inspecting graph(g), you can see that g is continuous everywhere, but g' is not defined at q.

By inspecting graph(h) in figure a below, you can see that h' is continuous, but you may have a hard time seeing the point where h'' is not defined.



The function h is defined by

$$h(x) = \begin{cases} x^2 - \frac{5}{2}x + 2 & \text{if } 0 \le x \le \frac{3}{2}. \\ \frac{1}{2}x^2 - x + \frac{7}{8} & \text{if } \frac{3}{2} < x \le 2. \end{cases} \tag{15.34}$$

so $h''(x) = 2$ for $0 < x < \frac{3}{2}$, and $h''(x) = 1$ for $\frac{3}{2} < x < 2$, and $h''(\frac{3}{2})$ is not defined. We constructed h by pasting together two parabolas. Figure b shows the two parabolas, one having a second derivative equal to 1, and the other having second derivative equal to 2.

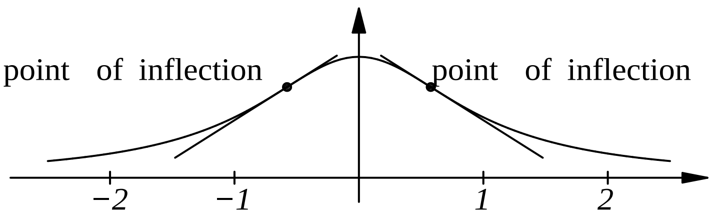
15.35 Exercise. Let h be the function described in formula (15.34). Draw graphs of h' and h''.

15.36 Entertainment (Discontinuous derivative problem.) There exists a function f such that f is differentiable everywhere on $\mathbf{R}$, but f' is discontinuous somewhere. Find such a function.

15.37 Exercise. Let $f(x) = x^4$. Show that $f''(0) = 0$, but 0 is not a point of inflection for f. Explain why this result does not contradict theorem 15.32

15.38 Example. Let
$$f(x) = \frac{1}{1+x^2}.$$
Then
$$f'(x) = \frac{-2x}{(1+x^2)^2},$$
and
$$f''(x) = \frac{(1+x^2)^2(-2) - (-2x)(2(1+x^2)(2x))}{(1+x^2)^4} = \frac{2(3x^2-1)}{(1+x^2)^3}.$$
Thus the only critical point for f is 0. Also,
$$(f'(x) > 0 \iff x < 0) \text{ and } (f'(x) < 0 \iff x > 0),$$
so f is increasing on $(-\infty, 0)$ and is decreasing on $(0, \infty)$. Thus f has a maximum at 0, and f has no minima.

We see that $f''(x) = 0 \iff x^2 = \frac{1}{3}$, and moreover
$$(f''(x) < 0) \iff x \in \left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right),$$
so f spills water over the interval $\Big(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\Big)$, and f holds water over each of the intervals $\Big(-\infty, -\sqrt{\frac{1}{3}}\Big)$ and $\Big(\sqrt{\frac{1}{3}}, \infty\Big)$. Thus f has points of inflection at $\pm\sqrt{\frac{1}{3}}$. We can use all of this information to make a reasonable sketch of the graph of f. Note that $f(x) > 0$ for all x, $f(0) = 1$, and $f\Big(\pm\sqrt{\frac{1}{3}}\Big) = \frac{3}{4}$, and $\sqrt{\frac{1}{3}}$ is approximately 0.58.



$$\left\{ y = \frac{1}{1+x^2} \right\}$$

15.39 Exercise. Discuss the graphs of the following functions. Make use of all the information that you can get by looking at the functions and their first two derivatives.

a) $f(x) = 5x^4 - 4x^5$.
b) $G(x) = 5x^3 - 3x^5$.
c) $H(x) = e^{-\frac{1}{x^2}}$.