Chapter 13

Applications

13.1 Curve Sketching

13.1 Example. Let $f(x) = \dfrac{x^3}{1-x^2}$. Here $\mathrm{dom}(f) = \mathbf{R} \setminus \{\pm 1\}$ and f is an odd function. We have

$$f'(x) = \frac{(1-x^2)3x^2 - x^3(-2x)}{(1-x^2)^2} = \frac{3x^2 - x^4}{(1-x^2)^2} = \frac{x^2(3-x^2)}{(1-x^2)^2}.$$

From this we see that the critical set for f is $\{0, \sqrt{3}, -\sqrt{3}\}$. We can determine the sign of $f'(x)$ by looking at the signs of its factors: Since f is odd, I will consider only points where $x > 0$.

	$0 < x < 1$	$1 < x < \sqrt{3}$	$\sqrt{3} < x$
$\frac{x^2}{(1-x^2)^2}$	$+$	$+$	$+$
$3 - x^2$	$+$	$+$	$-$
$f'(x)$	$+$	$+$	$-$

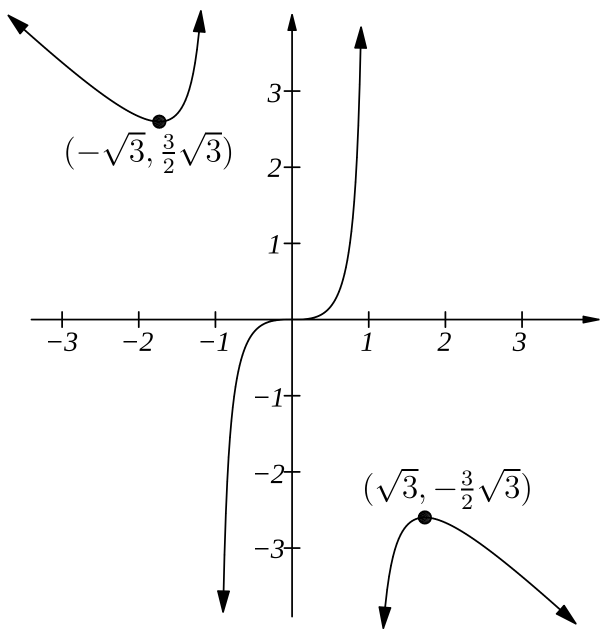
Thus f is strictly increasing on $(0,1)$ and on $(1,\sqrt{3})$, and f is strictly decreasing on $(\sqrt{3},\infty)$. Also

$$f(\sqrt{3}) = \frac{3\sqrt{3}}{1-3} = -\frac{3}{2}\sqrt{3} \text{ and } f(0) = 0.$$

We see that $|f(x)|$ is unbounded on any interval $(1-\delta, 1)$ or $(1, 1+\delta)$, since the numerator of the fraction is near to 1, and the denominator is near to 0 on these intervals. Also

$$f(x) = \frac{x^3}{1-x^2} = x\Big(\frac{x^2}{1-x^2}\Big) = x\Big(\frac{1}{-1+\frac{1}{x^2}}\Big),$$

so $|f(x)|$ is large when x is large. ($f(x)$ is the product of x and a number near to -1.) Using this information we can make a reasonable sketch of the graph of f.



Here f has a local maximum at $\sqrt{3}$ and a local minimum at $-\sqrt{3}$. It has no global extreme points.

13.2 Definition (Infinite limits.) Let $\{x_n\}$ be a real sequence. We say

$$\lim\{x_n\} = +\infty \text{ or } \{x_n\} \to +\infty$$

if for every $B \in \mathbf{R}$ there is an $N \in \mathbf{Z}^+$ such that for all $n \in \mathbf{Z}_{\geq N}$ $\Big(x_n > B\Big)$.

We say

$$\lim\{x_n\} = -\infty \text{ or } \{x_n\} \to -\infty$$

if for every $B \in \mathbf{R}$ there is an $N \in \mathbf{Z}^+$ such that for all $n \in \mathbf{Z}_{\geq N}$ $\big(x_n < B\big)$. Let f be a real valued function such that $\text{dom}(f) \subset \mathbf{R}$, and let $a \in \mathbf{R}$. We say

$$\lim_{x \to a^+} f(x) = +\infty$$

if $\text{dom}(f)$ contains an interval $(a, a+\epsilon)$ and for every sequence $\{x_n\}$ in $\text{dom}(f) \cap (a, \infty)$

$$(\{x_n\} \to a) \implies \big(\{f(x_n)\} \to +\infty\big).$$

We say

$$\lim_{x \to a^-} f(x) = +\infty$$

if $\text{dom}(f)$ contains an interval $(a-\epsilon, a)$ and for every sequence $\{x_n\}$ in $\text{dom}(f) \cap (-\infty, a)$

$$(\{x_n\} \to a) \implies \big(\{f(x_n)\} \to +\infty\big).$$

Similar definitions can be made for

$$\lim_{x \to a^+} f(x) = -\infty, \ \lim_{x \to a^-} f(x) = -\infty.$$

We say $\lim\limits_{x \to +\infty} f(x) = +\infty$ if $\text{dom}(f)$ contains some interval (a, ∞) and for every sequence $\{x_n\}$ in $\text{dom}(f)$

$$\{x_n\} \to +\infty \implies \{f(x_n)\} \to +\infty.$$

Similarly if $c \in \mathbf{R}$ we can define

$$\lim_{x \to +\infty} f(x) = -\infty, \ \lim_{x \to +\infty} f(x) = c, \ \lim_{x \to -\infty} f(x) = +\infty, \text{ etc.}$$

13.3 Example. If f is the function in the previous example (i.e. $f(x) = \dfrac{x^3}{1-x^2}$) then

$$\begin{aligned}
\lim_{x \to 1^+} f(x) &= -\infty, \\
\lim_{x \to 1^-} f(x) &= +\infty, \\
\lim_{x \to +\infty} f(x) &= -\infty,
\end{aligned}$$

and

$$\lim_{x \to -\infty} f(x) = +\infty.$$

Also,

$$\begin{aligned}
\lim_{x\to\infty}\frac{1}{x} &= 0,\\
\lim_{x\to 0^+}\frac{x}{|x|} &= 1,\\
\lim_{x\to 0^-}\frac{x}{|x|} &= -1,
\end{aligned}$$

and

$$\lim_{x\to+\infty}\frac{x^2+1}{x^2+3x} = \lim_{x\to+\infty}\frac{1+\frac{1}{x^2}}{1+\frac{3}{x}} = 1.$$

The situation here is very similar to the situation in the case of ordinary limits, and we will proceed without writing out detailed justifications.

13.4 Exercise. Write out definitions for

$$\Big(\lim_{x\to+\infty} f(x) = -\infty\Big) \text{ and for } \Big(\lim_{x\to a^-} f(x) = -\infty\Big).$$

13.5 Exercise. Find one function f satisfying all of the following conditions:

$$\begin{aligned}
\lim_{x\to+\infty} f(x) &= 3,\\
\lim_{x\to 3^+} f(x) &= +\infty,\\
\lim_{x\to 3^-} f(x) &= +\infty.
\end{aligned}$$

13.6 Example. Let $f(x) = \sin(2x) + 2\sin(x)$. Then $f(x+2\pi) = f(x)$ for all $x \in \mathbf{R}$, so I will restrict my attention to the interval $[-\pi,\pi]$. Also f is an odd function, so I will further restrict my attention to the interval $[0,\pi]$. Now

$$\begin{aligned}
f'(x) &= 2\cos 2x + 2\cos x = 2(2\cos^2 x - 1) + 2\cos x\\
&= 2(2\cos^2 x + \cos x - 1) = 2(2\cos x - 1)(\cos x + 1)\\
&= 4\Big(\cos x - \frac{1}{2}\Big)(\cos x + 1).
\end{aligned}$$

Hence x is a critical point for f if and only if $\cos x \in \Big\{\dfrac{1}{2}, -1\Big\}$. The critical points of f in $[0,\pi]$ are thus π and $\dfrac{\pi}{3}$, and the critical points in $[-\pi,\pi]$ are

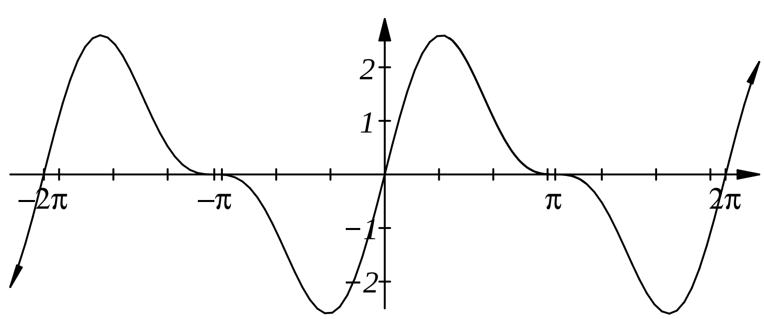
$\left\{-\pi, \pi, \frac{\pi}{3}, -\frac{\pi}{3}\right\}$. Now $f(\pi) = f(0) = 0$ and

$$f\left(\frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) + 2\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} = 2.6(\text{approximately}),$$

and $f\left(-\frac{\pi}{3}\right) = -f\left(\frac{\pi}{3}\right)$. Also note $f'(0) = 4$. Since f is continuous on $[-\pi, \pi]$, we know that f has a maximum and a minimum on this interval, and since $f(x + 2\pi) = f(x)$ for all $x \in \mathbf{R}$, the maximum (or minimum) of f on $[-\pi, \pi]$ will be a global maximum (or minimum) for f. Since f is differentiable everywhere, the extreme points are critical points and from our calculations f has a maximum at $\frac{\pi}{3}$ and a minimum at $-\frac{\pi}{3}$. I will now determine the sign of f' on $[0, \pi]$:

	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \pi$
$\cos x + 1$	$+$	$+$
$\cos x - \frac{1}{2}$	$+$	$-$
$f'(x)$	$+$	$-$

Thus f is strictly increasing on $\left(0, \frac{\pi}{3}\right)$ and f is strictly decreasing on $\left(\frac{\pi}{3}, \pi\right)$. We can now make a reasonable sketch for the graph of f.



13.7 Exercise. Sketch and discuss the graphs of the following functions. Mention all critical points and determine whether each critical point is a local or global maximum or minimum.

a) $f(x) = (1 - x^2)^2$.

b) $g(x) = \dfrac{x}{1+x^2}$.

c) $h(x) = x + \sin(x)$.

d) $k(x) = x\ln(x)$.

(The following remark may be helpful for determining $\lim\limits_{x\to 0} k(x)$. If $0 < t < 1$, then $\dfrac{1}{t} < \dfrac{1}{t^{\frac{3}{2}}}$. Hence if $0 < x < 1$, then

$$\begin{aligned} |\ln(x)| &= \left|\int_1^x \frac{1}{t}dt\right| = \int_x^1 \frac{1}{t}dt \leq \int_x^1 \frac{1}{t^{\frac{3}{2}}}dt \\ &= -\frac{2}{t^{\frac{1}{2}}}\bigg|_x^1 = 2\left(\frac{1}{\sqrt{x}} - 1\right) \leq \frac{2}{\sqrt{x}}. \end{aligned}$$

Thus,

$$|x\ln(x)| \leq 2\sqrt{x} \text{ for } 0 < x < 1).$$

13.2 Optimization Problems.

13.8 Example. A stick of length l is to be broken into four pieces of length s, s, t and t and the pieces are to be assembled to make a rectangle. How should s and t be chosen if the area of the rectangle is to be as large as possible? What is the area of this largest rectangle? Before doing the problem you should guess the answer. Your guess will probably be correct.

Let s be the length of one side of the rectangle. Then $2s + 2t = l$ so $t = \dfrac{l}{2} - s$; i.e., t is a function of s. Let $A(s)$ be the area of a rectangle with side s. Then

$$A(s) = st = s\Big(\frac{l}{2} - s\Big) = \frac{l}{2}s - s^2 \quad \text{for } 0 \leq s \leq \frac{l}{2}.$$

I include the endpoints for convenience; i.e., I consider rectangles with zero area to be admissible candidates for my answer. These clearly correspond to minimum area. Now

$$A'(s) = \frac{l}{2} - 2s$$

so A has only one critical point, namely $\frac{l}{4}$, and

$$A\Big(\frac{l}{4}\Big) = \frac{l}{2}\frac{l}{4} - \Big(\frac{l}{4}\Big)^2 = \frac{l^2}{16} = \Big(\frac{l}{4}\Big)^2.$$

Since A is continuous on $\Big[0, \frac{l}{2}\Big]$ we know that A has a maximum and a minimum, and since A is differentiable on $\Big(0, \frac{l}{2}\Big)$ the extreme points are a subset of $\Big\{0, \frac{l}{2}, \frac{l}{4}\Big\}$. Since $A(0) = A\Big(\frac{l}{2}\Big) = 0$ the maximal area is $\Big(\frac{l}{4}\Big)^2$; i.e., the maximal rectangle is a square. (As you probably guessed.)

This problem is solved by Euclid in completely geometrical terms [17, vol 1 page 382].

Euclid's proof when transformed from geometry to algebra becomes the following. Suppose in our problem $s \neq t$, say $s < t$. Since $s + t = \frac{l}{2}$, it follows that $s \leq \frac{l}{4} \leq t$ (if s and t were both less than $\frac{l}{4}$, we'd get a contradiction, and if they were both greater than $\frac{l}{4}$, we'd get a contradiction). Let r be defined by

$$s = \frac{l}{4} - r \text{ so } r \geq 0.$$

Then $t = \frac{l}{2} - s = \frac{l}{2} - \Big(\frac{l}{4} - r\Big) = \frac{l}{4} + r$ so

$$A(s) = st = \Big(\frac{l}{4} - r\Big)\Big(\frac{l}{4} + r\Big) = \Big(\frac{l}{4}\Big)^2 - r^2 = A\Big(\frac{l}{4}\Big) - r^2.$$

Hence, if $r > 0$, $A(s) < A\Big(\frac{l}{4}\Big)$ and to get a maximum we must have $r = 0$ and $s = \frac{l}{4}$. This proof requires knowing the answer ahead of time (but you probably were able to guess it). In any case, Euclid's argument is special, whereas our calculus proof applies in many situations.

Quadratic polynomials can be minimized (or maximized) without calculus by completing the square. For example, we have

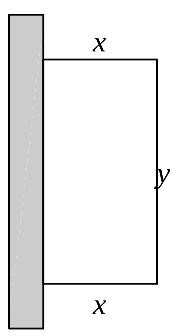
$$A(s) \quad = \quad -s^2 + \frac{l}{2}s$$

$$
\begin{aligned}
&= -\left(s^2 - \frac{l}{2}s + \left(\frac{l}{4}\right)^2\right) + \left(\frac{l}{4}\right)^2 \\
&= \left(\frac{l}{4}\right)^2 - \left(s - \frac{l}{4}\right)^2.
\end{aligned}
$$

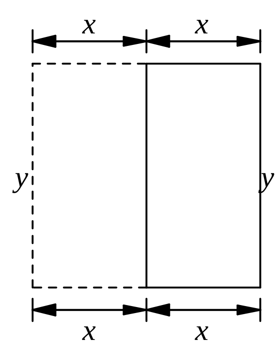
From this we can easily see that $A(s) \leq \left(\frac{l}{4}\right)^2$ for all s and equality holds only if $s = \frac{l}{4}$. This technique applies only to quadratic polynomials.

13.9 Example. Suppose I have 100 ft. of fence, and I want to fence off 3 sides of a rectangular garden, the fourth side of which lies against a wall and requires no fence (see the figure). What should the sides of the garden be if the area is to be as large as possible?

This is a straightforward problem, and in the next exercise you will do it by using calculus. Here I want to indicate how to do the problem without calculation. Imagine that the wall is a mirror, and that my fence is reflected in the wall.



2x+y=100

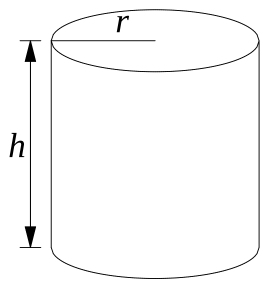


4x+2y=200

When I maximize the area of a garden with a rectangle of sides x and y, then I have maximized the area of a rectangle bounded by 200 feet of fence (on four sides) with sides y and $2x$. From the previous problem the answer to this problem is a square with $y = 2x = 50$. Hence, the answer to my original question is $y = 50$, $x = 25$. Often optimization problems have solutions that can be guessed on the basis of symmetry. You should try to guess answers to these problems before doing the calculations.

13.10 Exercise. Verify my solution in the previous example by using calculus *and* by completing the square.

13.11 Example. I want to design a cylindrical can of radius r and height h with a volume of V_0 cubic feet (V_0 is a constant). How should I choose r and h if the amount of tin in the can is to be minimum?



Here I don't see any obvious guess to make for the answer.

I have

$$V_0 = \text{ volume of can } = \pi r^2 h,$$

so $h = \dfrac{V_0}{\pi r^2}$. Let $A(r)$ be the surface area of the can of radius r. Then

$$\begin{aligned} A(r) &= \text{ area of sides } + 2 \text{ (area of top)} \\ &= 2\pi r \cdot h + 2(\pi r^2) \\ &= 2\pi r \frac{V_0}{\pi r^2} + 2\pi r^2 \\ &= \frac{2V_0}{r} + 2\pi r^2. \end{aligned}$$

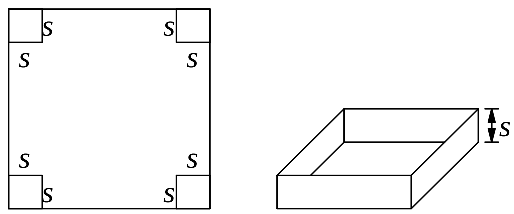
The domain of A is $\mathbf{R}^+$. It is clear that $\lim\limits_{r \to 0^+} A(r) = +\infty$, and $\lim\limits_{r \to +\infty} A(r) = +\infty$. Now $A'(r) = -\dfrac{2V_0}{r^2} + 4\pi r = \dfrac{4\pi}{r^2}\Big(r^3 - \dfrac{V_0}{2\pi}\Big)$. The only critical point for A is $r = \sqrt[3]{\dfrac{V_0}{2\pi}}$ (call this number r_0). Then $A'(r) = \dfrac{4\pi}{r^2}(r^3 - r_0^3)$. We see that $A'(r) < 0$ for $r \in (0, r_0)$ and $A'(r) > 0$ for $r \in (r_0, \infty)$ so A is decreasing on $(0, r_0]$ and A is increasing on $[r_0, \infty)$ and thus A has a minimum at r_0. The value of h corresponding to r_0 is

$$h = \frac{V_0}{\pi r_0^2} = \frac{V_0}{\pi \left(\frac{V_0}{2\pi}\right)^{2/3}} = \frac{2^{2/3}}{\pi^{1/3}} V_0^{1/3} = 2\Big(\frac{V_0^{1/3}}{2^{1/3}\pi^{1/3}}\Big) = 2r_0.$$

Thus the height of my can is equal to its diameter; i.e., the can will exactly fit into a cubical box.

In the following four exercises see if you can make a reasonable guess to the solutions before you use calculus to find them.

13.12 Exercise. A box (without a lid) is to be made by cutting 4 squares of side s from the corners of a $12'' \times 12''$ square, and folding up the corners as indicated in the figure.

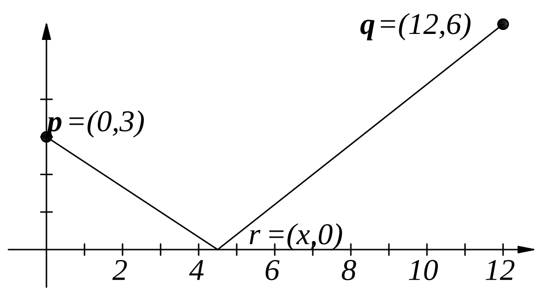


How should s be chosen to make the volume of the box as large as possible?

13.13 Exercise. A rectangular box with a square bottom and no lid is to be built having a volume of 256 cubic inches. What should the dimensions be, if the total surface area of the box is to be as small as possible?

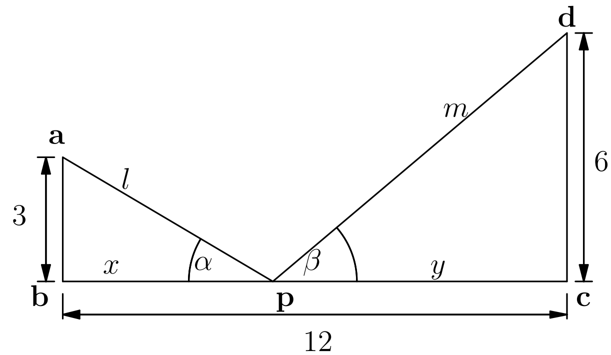
13.14 Exercise. Find the point(s) on the parabola whose equation is $y = x^2$ that are nearest to the point $\left(0, \frac{9}{2}\right)$.

13.15 Exercise. Let $\mathbf{p} = (0, 3)$ and let $\mathbf{q} = (12, 6)$. Find the point(s) $\mathbf{r}$ on the x-axis so that path from $\mathbf{p}$ to $\mathbf{r}$ to $\mathbf{q}$ is as short as possible; i.e., such that length([**pr**]) + length([**rq**]) is as short as possible.



You don't need to prove that the critical point(s) you find are actually minimum points.

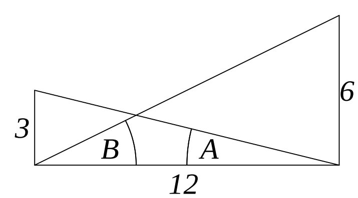
13.3 Rates of Change



Suppose in the given figure, I want to find the shortest path from **a** to a point **p** on the segment [**b c**] and back to **d**. Any such path will be uniquely defined by giving any one of the six numbers:

$$\begin{array}{ll}
x = \text{ dist}(\mathbf{b}, \mathbf{p}), & 0 \le x \le 12. \\
y = \text{ dist}(\mathbf{p}, \mathbf{c}), & 0 \le y \le 12 \\
l = \text{ dist}(\mathbf{a}, \mathbf{p}), & 3 \le l \le \sqrt{3^2 + 12^2}. \\
m = \text{ dist}(\mathbf{d}, \mathbf{p}), & 6 \le m \le \sqrt{6^2 + 12^2}. \\
\alpha = \angle \mathbf{apb}, & A \le \alpha \le \frac{\pi}{2}. \\
\beta = \angle \mathbf{dpc}, & B \le \beta \le \frac{\pi}{2}.
\end{array}$$

Here A, B are as shown in the figure below:



$$\frac{3}{12} = \frac{\sin(A)}{\cos(A)} = \tan(A) \text{ and } \frac{6}{12} = \frac{\sin(B)}{\cos(B)} = \tan(B).$$

For a given point **p**, any of the six numbers is a function of any of the others. For example, we have l is a function of x

$$l = \sqrt{x^2 + 9},$$

and l is a function of m, since for $x \in [0,12]$ and $y \in [0,12]$ we have

$$m^2 = (12-x)^2 + 36 \implies 12 - x = \sqrt{m^2 - 36}$$
$$\implies 12 - \sqrt{m^2-36} = x$$
$$\implies l = \sqrt{\left(12 - \sqrt{m^2-36}\right)^2 + 9}.$$

Also l is a function of α, since by similar triangles $\dfrac{\sin(\alpha)}{1} = \dfrac{3}{l}$ and hence

$$l = \frac{3}{\sin(\alpha)} = 3\csc(\alpha).$$

We have

$$\frac{dl}{dx} = \frac{1}{2}\frac{2x}{\sqrt{x^2+9}} = \frac{x}{\sqrt{x^2+9}}$$

and

$$\frac{dl}{d\alpha} = -3\csc(\alpha)\cot(\alpha).$$

I refer to $\dfrac{dl}{dx}$ as the *rate of change of* l *with respect to* x and to $\dfrac{dl}{d\alpha}$ as the *rate of change of* l *with respect to* α. Note that the "l"'s in "$\dfrac{dl}{dx}$" and "$\dfrac{dl}{d\alpha}$" represent different functions. In the first case $l(x) = \sqrt{x^2+9}$ and in the second case $l(\alpha) = 3\csc\alpha$. Here $\dfrac{dl}{dx}$ is positive, indicating that l increases when x increases, and $\dfrac{dl}{d\alpha}$ is negative, indicating that l decreases when α increases.

I want to find the path for which $l+m$ is shortest; i.e., I want to find the minimum value of $l+m$. I can think of l and m as being functions of x, and then the minimum value will occur when $\dfrac{d}{dx}(l+m) = 0$; i.e.,

$$\frac{dl}{dx} + \frac{dm}{dx} = 0. \tag{13.16}$$

Now $l^2 = x^2 + 9$, so $2l \cdot \dfrac{dl}{dx} = 2 \cdot x$; i.e.,

$$\frac{dl}{dx} = \frac{x}{l} = \cos\alpha,$$

and $m^2 = (12-x)^2 + 6^2$, so $2m\dfrac{dm}{dx} = 2(12-x)(-1)$, i.e.,

$$\frac{dm}{dx} = -\frac{(12-x)}{m} = -\frac{y}{m} = -\cos\beta.$$

Equation (13.16) thus says that for the minimum path $\cos\alpha - \cos\beta = 0$; i.e., $\cos\alpha = \cos\beta$, and hence $\alpha = \beta$. Thus the minimizing path satisfies the reflection condition, *angle of incidence equals angle of reflection.* Hence the minimizing triangle will make $\triangle\mathbf{bpa}$ and $\triangle\mathbf{cpd}$ similar, and will satisfy

$$\frac{6}{y} = \frac{3}{x} \text{ and } x + y = 12,$$

so

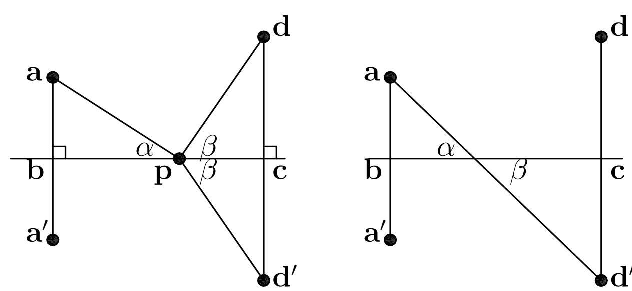
$$6x = 3y = 3(12-x) = 36 - 3x$$

or

$$9x = 36 \text{ so } x = 4 \text{ and } y = 8.$$

This example was done pretty much as Leibniz would have done it. You should compare the solution given here to your solution of exercise 13.15.

The problem in the last example was solved by Heron (date uncertain, sometime between 250 BC and 150 AD) as follows[26, page 353]. Imagine the line $[\mathbf{bc}]$ to be a mirror. Let $\mathbf{a}'$ and $\mathbf{d}'$ denote the images of $\mathbf{a}$ and $\mathbf{d}$ in the mirror,



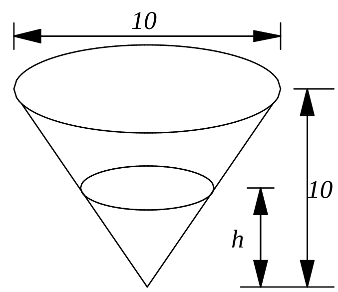
i.e. $[\mathbf{aa}']$ and $[\mathbf{dd}']$ are perpendicular to $[\mathbf{bc}]$ and $\text{dist}(\mathbf{a},\mathbf{b}) = \text{dist}(\mathbf{a}',\mathbf{b})$, $\text{dist}(\mathbf{d},\mathbf{c}) = \text{dist}(\mathbf{d}',\mathbf{c})$. Consider any path $\mathbf{apd}$ going from $\mathbf{a}$ to a point $\mathbf{p}$ on the mirror, and then to $\mathbf{d}$. Then triangle($\mathbf{pcd}$) and triangle($\mathbf{pcd}'$) are congruent, and hence

$$\text{dist}(\mathbf{p},\mathbf{d}) = \text{dist}(\mathbf{p},\mathbf{d}').$$

and hence the paths **apd** and **apd**$'$ have equal lengths. Now the shortest path **apd**$'$ is a straight line, which makes the angles α and β are vertical angles, which are equal. Hence the shortest path makes the angle of incidence equal to the angle of reflection, as we found above by calculus.

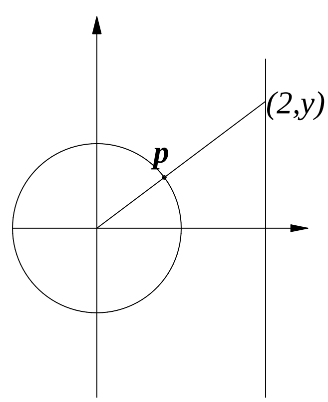
Remark: We can think of velocity as being rate of change of position with respect to time.

13.17 Exercise. Consider a conical tank in the shape of a right circular cone with altitude $10'$ and diameter $10'$ as shown in the figure.



Water flows into the tank at a constant rate of 10 cubic ft./minute. Let h denote the height of the water in the tank at a given time t. Find the rate of change of h with respect to t. What is this rate when the height of the water is $5'$? What can you say about $\dfrac{dh}{dt}$ when h is nearly zero?

13.18 Exercise. A particle **p** moves on the rim of a wheel of radius 1 that rotates about the origin at constant angular speed ω, so that at time t it is at the point $\big(\cos(\omega t), \sin(\omega t)\big)$. A light at the origin causes **p** to cast a shadow at the point $(2, y)$ on a wall two feet from the center of the wheel.



Find the rate of change of y with respect to time. You should ignore the speed of light, i.e. ignore the time it takes light to travel from the origin to the wall.