Chapter 12

Extreme Values of Functions

12.1 Continuity

12.1 Definition (Continuity at a point.) Let f be a real valued function such that $\text{dom}(f) \subset \mathbf{R}$. Let $a \in \text{dom}(f)$. We say that f *is continuous at* a if and only if

$$\lim_{x \to a} f(x) = f(a).$$

Remark: According to this definition, in order for f to be continuous at a we must have

$$a \in \text{dom}(f)$$

and

$$a \text{ is approachable from } \text{dom}(f).$$

The second condition is often not included in the definition of continuity, so this definition does not quite correspond to the usual definition.

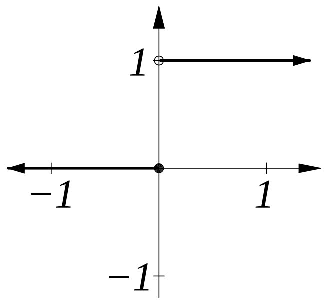
Remark: The method we will usually use to show that a function f is *not* continuous at a point a, is to find a sequence $\{x_n\}$ in $\text{dom}(f) \setminus \{a\}$ such that $\{x_n\} \to a$, but $\{f(x_n\}$ either diverges or converges to a value different from $f(a)$.

12.2 Definition (Continuity on a set.) Let f be a real valued function such that $\text{domain}(f) \subset \mathbf{R}$, and let S be a subset of $\text{domain}(f)$. We say that f *is continuous on* S if f is continuous at every point in S. We say that f *is continuous* if f is continuous at every point of $\text{domain}(f)$.

12.3 Example (sin, cos, ln and power functions are continuous.) We proved in lemma 11.17 that a function is continuous at every point at which it is differentiable. (You should now check the proof of that lemma to see that we did prove this.) Hence sin, cos, ln, and x^n (for $n \in \mathbf{Z}$) are all continuous on their domains, and if $r \in \mathbf{Q} \setminus \mathbf{Z}$, then x^r is continuous on $\mathbf{R}^+$.

12.4 Example. Let

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$



Then f is not continuous at 0. For the sequence $\{\frac{1}{n}\}$ converges to 0, but $\{f(\frac{1}{n})\} = \{1\} \to 1 \neq f(0)$.

Our limit rules all give rise to theorems about continuous functions.

12.5 Theorem (Properties of continuous functions.) *Let f, g be real valued functions with $\text{dom}(f) \subset \mathbf{R}$, $\text{dom}(g) \subset \mathbf{R}$, and let $c, a \in \mathbf{R}$. If f and g are continuous at a and if a is approachable from $\text{dom}(f) \cap \text{dom}(g)$, then $f+g$, $f-g$, fg, and cf are continuous at a. If in addition, $g(a) \neq 0$ then $\frac{f}{g}$ is also continuous at a.*

Proof: Suppose f and g are continuous at a, and a is approachable from $\text{dom}(f) \cap \text{dom}(g)$. Then

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a).$$

By the sum rule for limits (theorem 10.15) it follows that

$$\begin{aligned} \lim_{x \to a}(f+g)(x) &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \\ &= f(a) + g(a) = (f+g)(a). \end{aligned}$$

Thus $f+g$ is continuous at a. The proofs of the other parts of the theorem are similar.

12.6 Example (An everywhere discontinuous function.) Let D be the example of a non-integrable function defined in equation (8.37). Then D is not continuous at any point of $[0,1]$. Recall

$$D(x)=\begin{cases}1 & \text{if } x\in S\\ 0 & \text{if } x\neq S\end{cases}$$

where S is a subset of $[0,1]$ such that every subinterval of $[0,1]$ of positive length contains a point in S and a point not in S. Let $x\in[0,1]$.

Case 1. If $x\in S$ we can find a sequence of points $\{t_n\}$ in $[0,1]\setminus S$ such that $\{t_n\}\to x$. Then

$$\{D(t_n)\}=\{0\}\to 0\neq D(x)$$

so D is not continuous at x.

Case 2. If $x\notin S$ we can find a sequence of points $\{s_n\}$ in S such that $\{s_n\}\to x$. Then

$$\{D(s_n)\}=\{1\}\to 1\neq D(x)$$

so D is not continuous at x.

12.7 Example. Let

$$h(x)=\sqrt{x} \text{ for } x\in\mathbf{R}_{\geq 0}.$$

I claim that h is continuous. We know that h is differentiable on $\mathbf{R}^+$, so H is continuous at each point of $\mathbf{R}^+$. In example 10.13 we showed that $\lim\limits_{x\to 0}h(x)=0=h(0)$ so h is also continuous at 0.

12.8 Example. Let

$$\begin{aligned}f(x)&=-x^2,\\ g(x)&=\sqrt{x}.\end{aligned}$$

Then f and g are both continuous functions. Now

$$(g\circ f)(x)=\sqrt{-x^2}$$

and hence $g\circ f$ is not continuous. The domain of $g\circ f$ contains just one point, and that point is not approachable from $\operatorname{dom}(g\circ f)$.

12.9 Theorem (Continuity of compositions.) *Let f, g be functions with domains contained in $\mathbf{R}$ and let $a \in \mathbf{R}$. Suppose that f is continuous at a and g is continuous at $f(a)$. Then $g \circ f$ is continuous at a,* provided that a is approachable from $\mathrm{dom}(g \circ f)$.

Proof: Suppose f is continuous at a and g is continuous at $f(a)$, and a is approachable from $\mathrm{dom}(g \circ f)$. Let $\{x_n\}$ be a sequence in $\mathrm{dom}(g \circ f) \setminus \{a\}$ such that $\{x_n\} \to a$. Then $\{f(x_n)\} \to f(a)$ since f is continuous at a. Hence $\{g\big(f(x_n)\big)\} \to g\big(f(a)\big)$ since g is continuous at $f(a)$; i.e.,

$$\{(g \circ f)(x_n)\} \to (g \circ f)(a).$$

Hence $g \circ f$ is continuous at a.

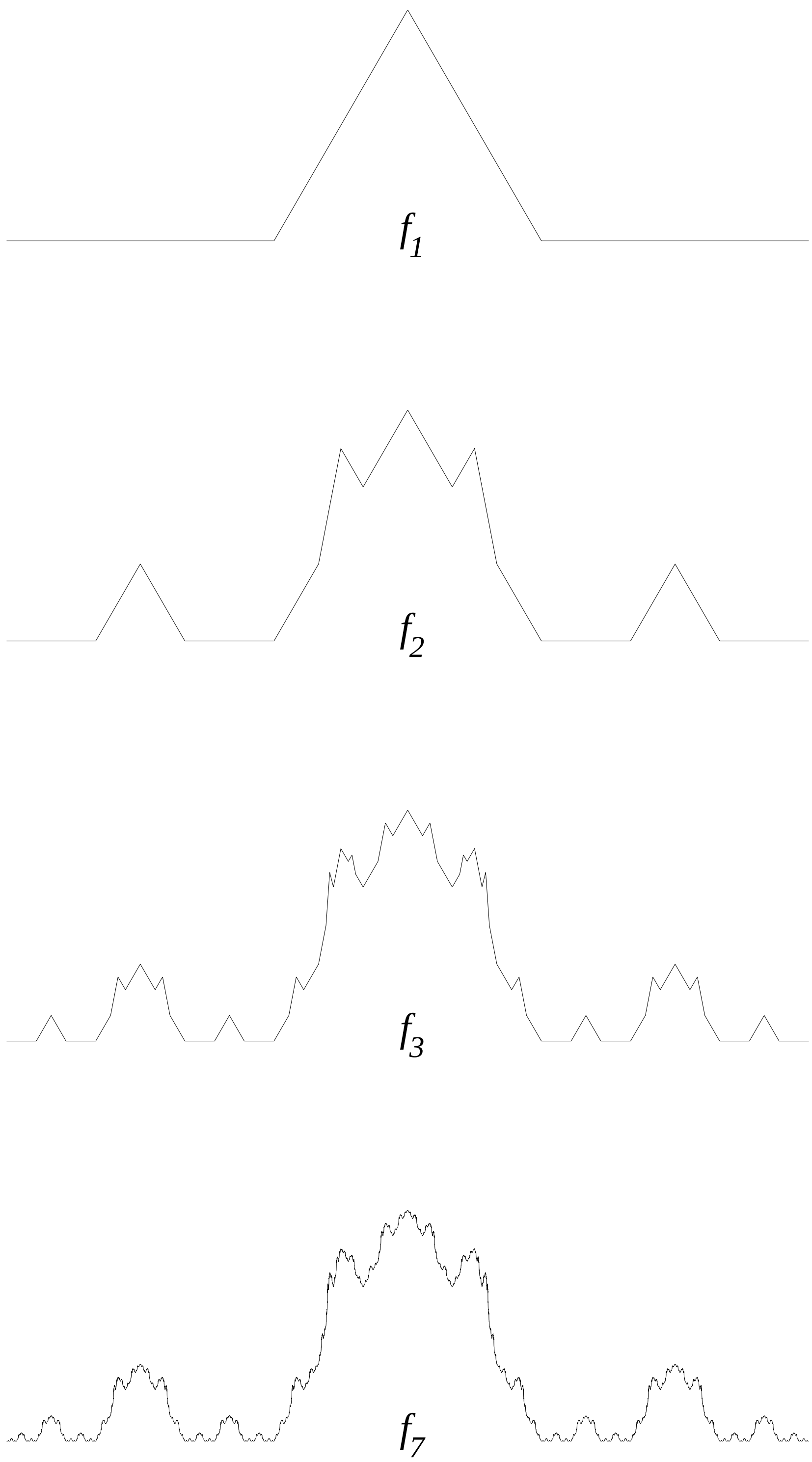
12.2 *A Nowhere Differentiable Continuous Function.

We will now give an example of a function f that is continuous at every point of $[0, 1]$ and differentiable at no point of $[0, 1]$. The first published example of such a function appeared in 1874 and was due to Karl Weierstrass(1815-1897) [29, page 976]. The example described below is due to Helga von Koch (1870-1924), and is a slightly modified version of Koch's snowflake. From the discussion in section 2.6, it is not really clear what we would mean by the *perimeter* of a snowflake, but it is pretty clear that whatever the perimeter might be, it is not the graph of a function. However, a slight modification of Koch's construction yields an everywhere continuous but nowhere differentiable *function.*

We will construct a sequence $\{f_n\}$ of functions on $[0, 1]$. The graph of f_n will be a polygonal line with 4^{n-1} segments. We set

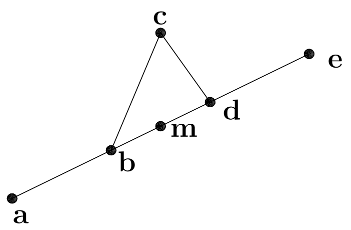
$$f_1(x) = 0 \quad \text{for} \quad 0 \le x \le 1$$

so that the graph of f_1 is the line segment from $(0, 0)$ to $(0, 1)$.



Approximations to a nowhere differentiable function

In general the graph of f_{n+1} is obtained from the graph of f_n by replacing each segment $[\mathbf{ae}]$ in the graph of f_n by four segments $[\mathbf{ab}], [\mathbf{bc}], [\mathbf{cd}]$, and $[\mathbf{de}]$ constructed according to the following three rules:



i) The points $\mathbf{b}$ and $\mathbf{d}$ trisect the segment $[\mathbf{ae}]$.

ii) The point $\mathbf{c}$ lies above the midpoint $\mathbf{m}$ of $[\mathbf{ae}]$.

iii) $\text{distance}(\mathbf{m}, \mathbf{c}) = \dfrac{\sqrt{3}}{2} \text{distance}(\mathbf{b}, \mathbf{d})$.

The graphs of f_2, f_3, f_4 and f_7 are shown on page 260. It can be shown that for each $x \in [0, 1]$ the sequence $\{f_n(x)\}$ converges. Define f on $[0, 1]$ by

$$f(x) = \lim\{f_n(x)\} \text{ for all } x \in [0, 1].$$

It turns out that f is continuous on $[0, 1]$ and differentiable nowhere on $[0, 1]$. A proof of this can be found in [31, page 168].

The function f provides us with an example of a continuous function that is not piecewise monotonic over any interval.

12.3 Maxima and Minima

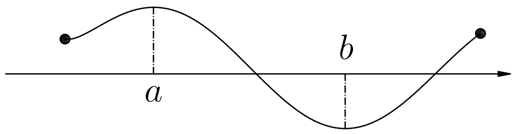
12.10 Definition (Maximum, minimum, extreme points.) Let A be a set, let $f: A \to \mathbf{R}$ and let $a \in A$. We say that f has a *maximum* at a if

$$f(a) \geq f(x) \text{ for all } x \in A,$$

and we say that f has a *minimum* at a if

$$f(a) \leq f(x) \text{ for all } x \in A.$$

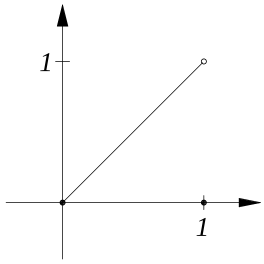
Points a where f has a maximum or a minimum are called *extreme points of* f.



f has a maximum at a and a minimum at b

12.11 Example. Let $f\colon [0,1] \to \mathbf{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } x = 1. \end{cases} \tag{12.12}$$

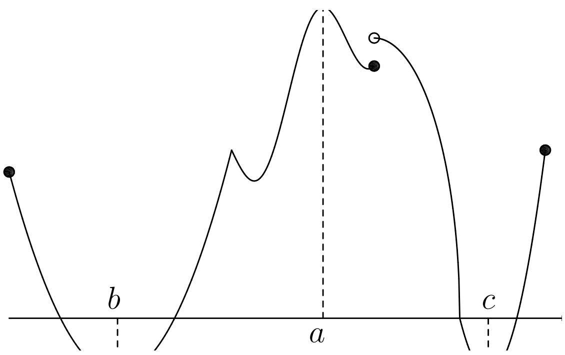


$\{y = f(x)\}$

Then f has a minimum at 0 and at 1, but f has no maximum. To see that f has no maximum, observe that if $a \in [0,1)$ then $\dfrac{1+a}{2} \in [0,1)$ and

$$f(\frac{1+a}{2}) = \frac{1+a}{2} > \frac{a+a}{2} = a = f(a).$$

If g is the function whose graph is shown, then g has a maximum at a, and g has minimums at b and c.



12.13 Assumption (Extreme value property.) If f is a continuous function on the interval $[a, b]$, then f has a maximum and a minimum on $[a, b]$.

The extreme value property is another assumption that is really a theorem, (although the proof requires yet another assumption, namely *completeness* of the real numbers.)

The following exercise shows that all of the hypotheses of the extreme value property are necessary.

12.14 Exercise.

a) Give an example of a continuous function f on $(0, 1)$ such that f has no maximum on $(0, 1)$.

b) Give an example of a bounded continuous function g on the closed interval $[0, \infty)$, such that g has no maximum on $[0, \infty)$

c) Give an example of a function h on $[0, 1]$ such that h has no maximum on $[0, 1]$.

d) Give an example of a continuous function k on $[0, \infty)$ that has neither a maximum nor a minimum on $[0, \infty)$, or else explain why no such function exists.

12.15 Exercise.

a) Show that every continuous function from an interval $[a, b]$ to $\mathbf{R}$ is bounded. (Hint: Use the extreme value property,)

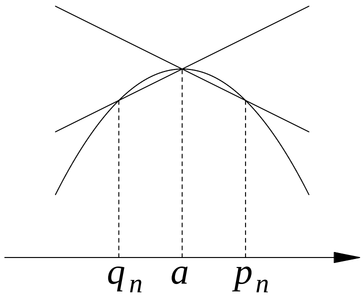
b) Is it true that every continuous function from an open interval (a, b) to $\mathbf{R}$ is bounded?

c) Give an example of a function from $[0, 1]$ to $\mathbf{R}$ that is not bounded.

12.16 Definition (Critical point, critical set.) Let f be a real valued function such that $\text{dom}(f) \subset \mathbf{R}$. A point $a \in \text{dom}(f)$ is called a *critical point for* f if $f'(a) = 0$. The set of critical points for f is the *critical set* for f. The points x in the critical set for f correspond to points $\big(x, f(x)\big)$ where the graph of f has a horizontal tangent.

12.17 Theorem (Critical point theorem I.) *Let f be a real valued function with* $\mathrm{dom}(f) \subset \mathbf{R}$. *Let $a \in \mathbf{R}$. If f has a maximum (or a minimum) at a, and f is differentiable at a, then $f'(a) = 0$.*

Proof: We will consider only the case where f has a maximum. Suppose f has a maximum at a and f is differentiable at a. Then a is an interior point of $\mathrm{dom}(f)$ so we can find sequences $\{p_n\}$ and $\{q_n\}$ in $\mathrm{dom}(f) \setminus \{a\}$ such that $\{p_n\} \to a$, $\{q_n\} \to a$, $p_n > a$ for all $n \in \mathbf{Z}^+$, and $q_n < a$ for all $n \in \mathbf{Z}^+$.



Since f has a maximum at a, we have $f(p_n) - f(a) \leq 0$ and $f(q_n) - f(a) \leq 0$ for all n. Hence

$$\frac{f(p_n) - f(a)}{p_n - a} \leq 0 \text{ and } \frac{f(q_n) - f(a)}{q_n - a} \geq 0 \text{ for all } n.$$

Hence by the inequality theorem for limits,

$$f'(a) = \lim\left\{\frac{f(p_n) - f(a)}{p_n - a}\right\} \leq 0 \text{ and } f'(a) = \lim\left\{\frac{f(q_n) - f(a)}{q_n - a}\right\} \geq 0.$$

It follows that $f'(a) = 0$. |||

12.18 Definition (Local maximum and minimum.) Let f be a real valued function whose domain is a subset of $\mathbf{R}$. Let $a \in \mathrm{dom}(f)$. We say that f has a *local maximum* at a if there is a positive number δ such that

$$f(a) \geq f(x) \text{ for all } x \in \mathrm{dom}(f) \cap (a - \delta, a + \delta),$$

and we say that f has a *local minimum* at a if there is a positive number δ such that

$$f(a) \leq f(x) \text{ for all } x \in \mathrm{dom}(f) \cap (a - \delta, a + \delta).$$

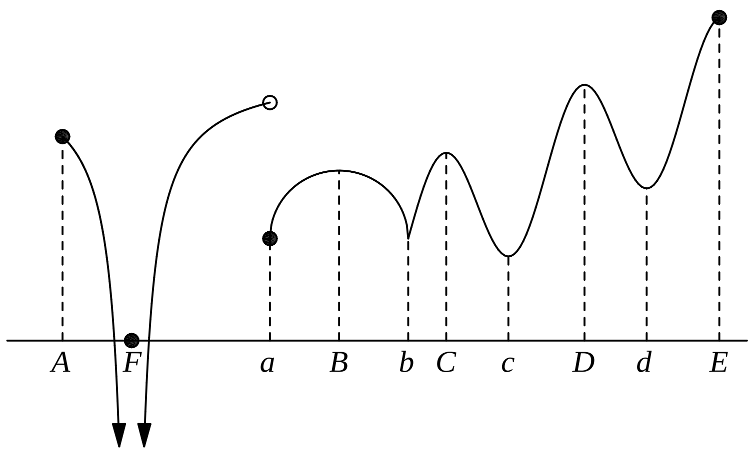
Sometimes we say that f has a *global maximum at a* to mean that f has a maximum at a, when we want to emphasize that we do not mean local maximum. If f has a local maximum or a local minimum at a we say f has a *local extreme point* at a.

12.19 Theorem (Critical point theorem II.) *Let f be a real valued function with* $\mathrm{dom}(f) \subset \mathbf{R}$*. Let $a \in \mathbf{R}$. If f has a local maximum or minimum at a, and f is differentiable at a, then $f'(a) = 0$.*

Proof: The proof is the same as the proof of theorem 12.17.

12.20 Examples. If f has a maximum at a, then f has a local maximum at a.

The function g whose graph is shown in the figure has local maxima at A, B, C, D, E, F and local minima at a, b, c, and d. It has a global maximum at E, and it has no global minimum.



From the critical point theorem, it follows that to investigate the extreme points of f, we should look at critical points, or at points where f is not differentiable (including endpoints of domain f).

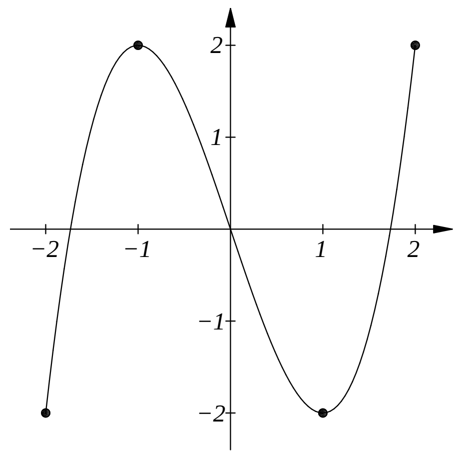
12.21 Example. Let $f(x) = x^3 - 3x$ for $-2 \le x \le 2$. Then f is differentiable everywhere on $\mathrm{dom}(f)$ except at 2 and -2. Hence, any local extreme points are critical points of f or are in $\{2, -2\}$. Now

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x - 1)(x + 1).$$

From this we see that the critical set for f is $\{-1, 1\}$. Since f is a continuous function on a closed interval $[-2, 2]$ we know that f has a maximum and a minimum on $[-2, 2]$. Now

$$f(-2) = -2,\ f(-1) = 2,\ f(1) = -2,\ f(2) = 2.$$

Hence f has global maxima at -1 and 2, and f has global minima at -2 and 1. The graph of f is shown.



$\{y = x^3 - 3x)\}$

12.22 Example. Let

$$f(x) = \frac{1}{1+x^2}.$$

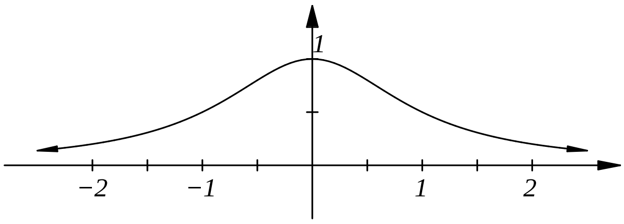
Here $\text{dom}(f) = \mathbf{R}$ and clearly $f(x) > 0$ for all x. I can see by inspection that f has a maximum at 0; i.e.,

$$f(x) = \frac{1}{1+x^2} \leq \frac{1}{1+0} = 1 = f(0) \text{ for all } x \in \mathbf{R}$$

I also see that $f(-x) = f(x)$, and that f is strictly decreasing on $\mathbf{R}^+$

$$0 < x < t \implies x^2 < t^2 \implies 1 + x^2 < 1 + t^2 \implies \frac{1}{1+x^2} > \frac{1}{1+t^2}$$

thus f has no local extreme points other than 0. Also $f(x)$ is very small when x is large. There is no point in calculating the critical points here because all the information about the extreme points is apparent without the calculation.



$\{y = \frac{1}{1+x^2}\}$

12.23 Exercise. Find and discuss all of the global and local extreme points for the following functions. Say whether the extreme points are maxima or minima, and whether they are global or local.

a) $f(x) = x^4 - x^2$ for $-2 \le x \le 2$.

b) $g(x) = 4x^3 - 3x^4$ for $-2 \le x \le 2$.

12.4 The Mean Value Theorem

12.24 Lemma (Rolle's Theorem) *Let a, b be real numbers with $a < b$ and let $f\colon [a, b] \to \mathbf{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = f(b)$. Then there is a point $c \in (a, b)$ such that $f'(c) = 0$.*

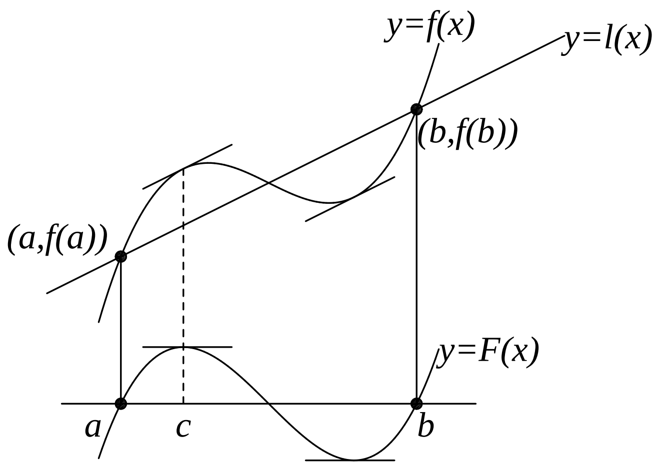
Proof: By the extreme value property, f has a maximum at some point $A \in [a, b]$. If $A \in (a, b)$, then $f'(A) = 0$ by the critical point theorem. Suppose $A \in \{a, b\}$. By the extreme value property, f has a minimum at some point $B \in [a, b]$. If $B \in (a, b)$ then $f'(B) = 0$ by the critical point theorem. If $B \in \{a, b\}$, then we have $\{A, B\} \subset \{a, b\}$ so $f(A) = f(B) = f(a) = f(b)$. Hence in this case the maximum value and the minimum value taken by f are equal, so $f(x) = f(a)$ for $x \in [a, b]$ so $f'(x) = 0$ for all $x \in (a, b)$. |||

Rolle's theorem is named after Michel Rolle (1652-1719). An English translation of Rolle's original statement and proof of the theorem can be found in [43, pages 253-260]. It takes a considerable effort to see any relation between what Rolle says and what our form of Rolle's theorem says.

12.25 Theorem (Mean value theorem.) *Let a, b be real numbers and let $f\colon [a,b] \to \mathbf{R}$ be a function that is continuous on $[a,b]$ and differentiable on (a,b). Then there is a point $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$; i.e., there is a point c where the slope of the tangent line is equal to the slope of the line joining $\big(a, f(a)\big)$ to $\big(b, f(b)\big)$.*

Proof: The equation of the line joining $\big(a, f(a)\big)$ to $\big(b, f(a)\big)$ is

$$y = l(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a).$$



Let

$$\begin{aligned} F(x) &= f(x) - l(x) \\ &= f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a). \end{aligned}$$

Then F is continuous on $[a,b]$ and differentiable on (a,b) and $F(a) = F(b) = 0$. By Rolle's theorem there is a point $c \in (a,b)$ where $F'(c) = 0$.

Now

$$F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a},$$

so

$$
\begin{aligned}
F'(c) = 0 \implies & f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \\
\implies & f'(c) = \frac{f(b)-f(a)}{b-a}. \ \|
\end{aligned}
$$

12.26 Corollary. *Let J be an interval in $\mathbf{R}$ and let $f\colon J \to \mathbf{R}$ be a function that is continuous on J and differentiable at the interior points of J. Then*

$$
\begin{aligned}
f'(x) = 0 \text{ for all } x \in \text{interior}(J) &\implies f \text{ is constant on } J. \\
f'(x) \le 0 \text{ for all } x \in \text{interior}(J) &\implies f \text{ is decreasing on } J. \\
f'(x) \ge 0 \text{ for all } x \in \text{interior}(J) &\implies f \text{ is increasing on } J. \\
f'(x) < 0 \text{ for all } x \in \text{interior}(J) &\implies f \text{ is strictly decreasing on } J. \\
f'(x) > 0 \text{ for all } x \in \text{interior}(J) &\implies f \text{ is strictly increasing on } J.
\end{aligned}
$$

Proof: I will prove the second assertion. Suppose $f'(x) \le 0$ for all $x \in \text{interior}(J)$. Let s, t be points in J with $s < t$. Then by the mean value theorem

$$f(t) - f(s) = f'(c)(t-s) \text{ for some } c \in (s,t).$$

Since $f'(c) \le 0$ and $(t-s) > 0$, we have $f(t) - f(s) = f'(c)(t-s) \le 0$; i.e., $f(t) \le f(s)$. Thus f is decreasing on J. $\|$

12.27 Exercise. Prove the first assertion of the previous corollary; i.e., prove that if f is continuous on an interval J, and $f'(x) = 0$ for all $x \in \text{interior}(J)$, then f is constant on J.

12.28 Definition (Antiderivative) Let f be a real valued function with $\text{dom}(f) \subset \mathbf{R}$. Let J be an interval such that $J \subset \text{dom}(f)$. A function F is an *antiderivative for* f on J if F is continuous on J and $F'(x) = f(x)$ for all x in the interior of J.

12.29 Examples. Since $\dfrac{d}{dx}(x^3+4) = 3x^2$, we see that $x^3 + 4$ is an antiderivative for $3x^2$. Since

$$\frac{d}{dx}(\cos^2(x)) = 2\cos(x)(-\sin(x)) = -2\sin(x)\cos(x),$$

and

$$\frac{d}{dx}(-\sin^2(x)) = -2\cdot\sin(x)\cos(x)$$

we see that $\cos^2$ and $-\sin^2$ are both antiderivatives for $-2\sin\cdot\cos$.

We will consider the problem of finding antiderivatives in chapter 17. Now I just want to make the following observation:

12.30 Theorem (Antiderivative theorem.) *Let f be a real valued function with* $\text{dom}(f) \subset \mathbf{R}$ *and let J be an interval with $J \subset \text{dom}(f)$. If F and G are two antiderivatives for f on J, then there is a number $c \in \mathbf{R}$ such that*

$$F(x) = G(x) + c \textit{ for all } x \in J.$$

12.31 Exercise. Prove the antiderivative theorem.

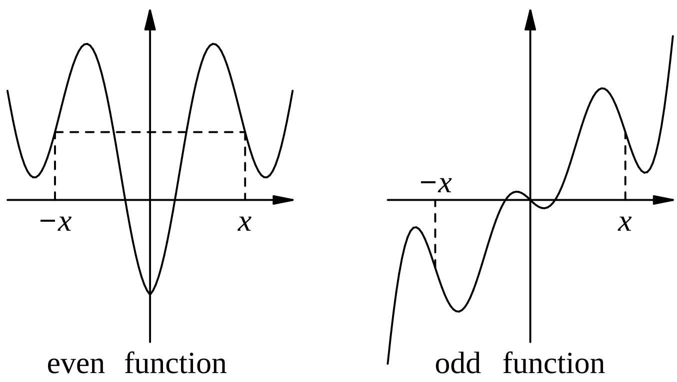
12.32 Definition (Even and odd functions.) A subset S of $\mathbf{R}$ is called *symmetric* if $(x \in S \implies -x \in S)$. A function f is said to be *even* if $\text{dom}(f)$ is a symmetric subset of $\mathbf{R}$ and

$$f(x) = f(-x) \text{ for all } x \in \text{dom}(f),$$

and f is said to be *odd* if $\text{dom}(f)$ is a symmetric subset of $\mathbf{R}$ and

$$f(x) = -f(-x) \text{ for all } x \in \text{dom}(f)$$

.



12.33 Example. If $n \in \mathbf{Z}^+$ and $f(x) = x^n$, then f is even if n is even, and f is odd if n is odd. Also cos is an even function and sin is an odd function, while ln is neither even or odd.

12.34 Example. If f is even, then $V\Big(\text{graph}(f)\Big) = \text{graph}(f)$ where V is the reflection about the vertical axis. If f is odd, then $R_\pi\Big(\text{graph}(f)\Big) = \text{graph}(f)$ where R_π is a rotation by π about the origin.

12.35 Exercise. Are there any functions that are both even and odd?

12.36 Exercise.

a) If f is an arbitrary even differentiable function, show that the derivative of f is odd.

b) If g is an arbitrary odd differentiable function, show that the derivative of g is even.