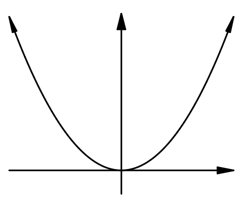
Chapter 10

Definition of the Derivative

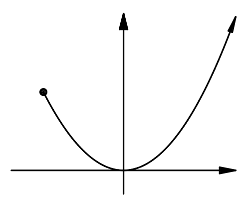
10.1 Velocity and Tangents

10.1 Notation. If $E_1(x, y)$ and $E_2(x, y)$ denote equations or inequalities in x and y, we will use the notation

$$\begin{aligned}\{E_1(x,y)\} &= \{(x,y) :\in \mathbf{R}^2 \colon E_1(x,y)\} \\ \{E_1(x,y); E_2(x,y)\} &= \{(x,y) \in \mathbf{R}^2 \colon E_1(x,y) \text{ and } E_2(x,y)\}.\end{aligned}$$



$\{y = x^2\}$



$\{y = x^2; x \geq 1\}$

In this section we will discuss the idea of *tangent to a curve* and the related idea of *velocity of a moving point*.

You probably have a pretty good intuitive idea of what is meant by the tangent to a curve, and you can see that the straight lines in figure a below represent tangent lines to curves.

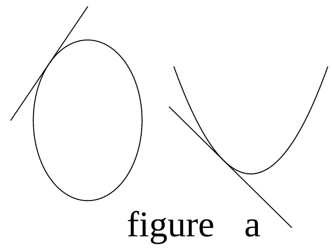
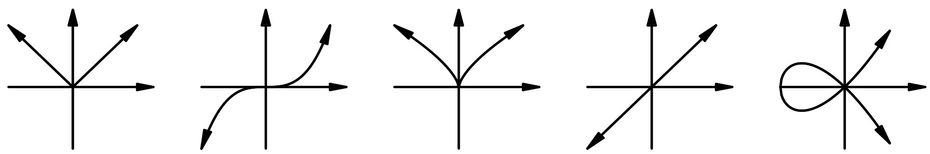


figure a

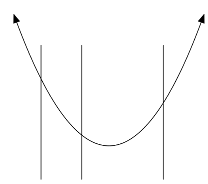
It may not be quite so clear what you would mean by the tangents to the curves in figure b at the point $(0,0)$.



$\{y=|x|\}$ $\{y=x^3\}$ $\{y^3=x^2\}$ $\{y=x\}$ $\{y^2=x^3+x^2\}$

figure b

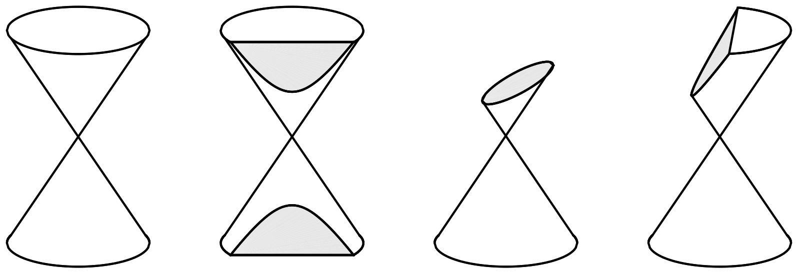
Euclid (fl. c. 300 B.C.) defined a tangent to a circle to be a line which touches the circle in exactly one point. This is a satisfactory definition of tangent to a circle, but it does not generalize to more complicated curves.



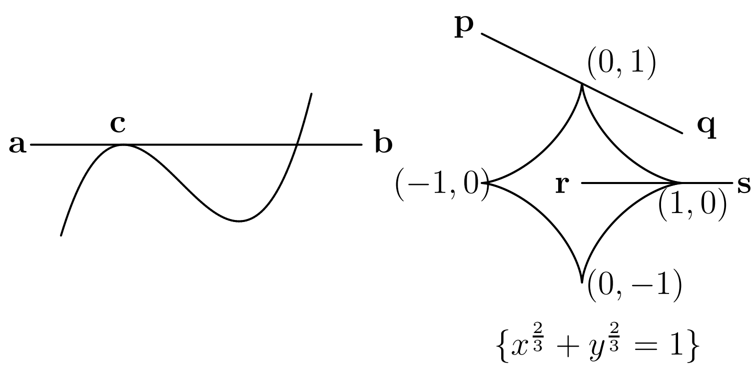
$\{y=x^2\}$

For example, every vertical line intersects the parabola $\{y=x^2\}$ in just one point, but no such line should be considered to be a tangent.

Apollonius (c 260-170 B.C.) defined a tangent to a conic section (i.e., an ellipse or hyperbola or parabola) to be a line that touches the section, but lies outside of the section. Apollonius considered these sections to be obtained by intersecting a cone with a plane, and points inside of the section were points in the cone.



This definition works well for conic sections, but for general curves, we have no notion of what points lie inside or outside a curve.

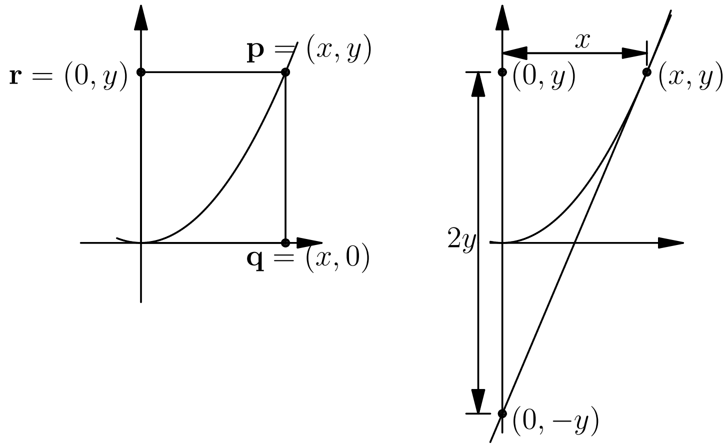


In the figure, the line **ab** ought to be tangent to the curve at **c**, but there is no reasonable sense in which the line lies outside the curve. On the other hand, it may not be clear whether **pq** (which lies outside the curve $\{x^{2/3} + y^{2/3} = 1\}$ is more of a tangent than the line **rs** which does not lie outside of it. Leibniz [33, page 276] said that

> to find a *tangent* means to draw a line that connects two points of the curve at an infinitely small distance, or the continued side of a polygon with an infinite number of angles, which for us takes the place of the *curve*.

From a modern point of view it is hard to make any sense out of this.

Here is a seventeenth century sort of argument for finding a tangent to the parabola whose equation is $y = x^2$.



Imagine a point **p** that is moving along the parabola $y = x^2$, so that at time t, **p** is at (x, y). (Here x and y are functions of t, but in the seventeenth century they were just flowing quantities.) Imagine a point **q** that moves along the x-axis so that **q** always lies under **p** and a point **r** moving along the y-axis so that **r** is always at the same height as **p**. Let $\dot{x}$ denote the velocity of **q** when **p** is at (x, y) and let $\dot{y}$ denote the velocity of **r** when **p** is at (x, y). Let o be a very small moment of time. At time o after **p** is at (x, y), **p** will be at $(x + o\dot{x}, y + o\dot{y})$ (i.e., x will have increased by an amount equal to the product of the time interval o and its velocity $\dot{x}$). Since **p** stays on the curve, we have

$$y + o\dot{y} = (x + o\dot{x})^2$$

or

$$y + o\dot{y} = x^2 + 2xo\dot{x} + o^2\dot{x}^2.$$

Since $y = x^2$, we get

$$o\dot{y} = 2xo\dot{x} + o^2\dot{x}^2 \tag{10.2}$$

or

$$\dot{y} = 2x\dot{x} + o\dot{x}^2 \tag{10.3}$$

Since we are interested in the velocities at the instant that **p** is at (x, y), we take $o = 0$, so

$$\dot{y} = 2x\dot{x}.$$

Hence when p is at (x, y), the vertical part of its velocity (i.e., $\dot{y}$) is $2x$ times the horizontal component of its velocity. Now the velocity should point in the direction of the curve; i.e., in the direction of the tangent, so the direction of the tangent at (x, y) should be in the direction of the diagonal of a box with

$$\text{vertical side } = 2x \times \text{ horizontal side.}$$

The tangent to the parabola at $(x, y) = (x, x^2)$ is the line joining (x, y) to $(0, -y)$, since in the figure the vertical component of the box is

$$2y = 2x^2 = (2x)x;$$

i.e., the vertical component is $2x$ times the horizontal component.

In *The Analyst: A Discourse Addressed to an Infidel Mathematician*[7, page 73], George Berkeley (1685-1753) criticizes the argument above, pointing out that when we divide by o in line (10.3) we must assume o is not zero, and then at the end we set o equal to 0.

> All which seems a most inconsistent way of arguing, and such as would not be allowed of in Divinity.

The technical concept of velocity is not a simple one. The idea of *uniform velocity* causes no problems: to quote Galileo (1564-1642):

> By steady or uniform motion, I mean one in which the distances traversed by the moving particle during any equal intervals of time, are themselves equal[21, page 154].

This definition applies to points moving in a straight line, or points moving on a circle, and it goes back to the Greek scientists. The problem of what is meant by velocity for a non-uniform motion, however, is not at all clear. The Greeks certainly realized that a freely falling body moves faster as it falls, but they had no language to describe the way in which velocity changes. Aristotle (384-322 B.C.) says

> there cannot be motion of motion or becoming of becoming or in general change of change[11, page 168].

It may not be clear what this means, but S. Bochner interprets this as saying that the notion of a second derivative (this is a technical term for the mathematical concept used to describe acceleration which we will discuss later) is a meaningless idea[11, page 167]. Even though we are in constant contact with non-uniformly moving bodies, our intuition about the way they move is not very good. In the *Dialogues Concerning Two New Sciences*, Salviati (representing Galileo) proposes the hypothesis that if a stone falls from rest, then it falls in such a way that "in any equal intervals of time whatever, equal increments of speed are given to it"[21, page 161].

In our language, the hypothesis is that the velocity $v(t)$ at time t satisfies

$$v(t) = kt \text{ for some constant } k.$$

Sagredo objects to this on the grounds that this would mean that the object begins to fall with zero speed "while our senses show us that a heavy falling body suddenly acquires great speed." (I believe Sagredo is right. Try dropping some bodies and observe how they begin to fall.) Salviati replies that this is what he thought at first, and explains how he came to change his mind.

Earlier, in 1604, Galileo had supposed that

$$v(x) = kx \text{ for some constant } k;$$

i.e., that in equal increments of distance the object gains equal increments of speed (which is false), and Descartes made the same error in 1618 [13, page 165]. Casual observation doesn't tell you much about falling stones.

10.4 Entertainment (Falling bodies.) Try to devise an experiment to support (or refute) Galileo's hypothesis that $v(t) = kt$, using materials available to Galileo; e.g., no stop watch. Galileo describes his experiments in [21, pages 160-180], and it makes very good reading.

10.2 Limits of Functions

Our definition of tangent to a curve is going to be based on the idea of *limit*. The word *limit* was used in mathematics long before the definition we will give was thought of. One finds statements like "The limit of a regular polygon when the number of sides becomes infinite, is a circle." Early definitions of limit often involved the ideas of time or motion. Our definition will be purely mathematical.

10.5 Definition (Interior points and approachable points.) Let S be a subset of $\mathbf{R}$. A point $x \in S$ is an *interior point* of S if there is some positive number ϵ such that the interval $(x - \epsilon, x + \epsilon)$ is a subset of S. A point $x \in R$

is an *approachable point from* S if there is some positive number ϵ such that either $(x-\epsilon, x) \subset S$ or $(x, x+\epsilon) \subset S$. (Without loss of generality we could replace "ϵ" in this definition by $\dfrac{1}{N}$ for some $N \in \mathbf{Z}^+$.)

Note that interior points of S must belong to S. Approachable points of S need not belong to S. Any interior point of S is approachable from S.

10.6 Example. If S is the open interval $(0, 1)$ then every point of S is an interior point of S. The points that are approachable from S are the points in the closed interval $[0, 1]$.

If T is the closed interval $[0, 1]$ then the points that are approachable from T are exactly the points in T, and the interior points of T are the points in the open interval $(0, 1)$.

10.7 Definition (Limit of a function.) Let f be a real valued function such that $\mathrm{dom}(f) \subset \mathbf{R}$. Let $a \in \mathbf{R}$ and let $L \in \mathbf{R}$. We say

$$\lim_{x\to a} f(x) = L \tag{10.8}$$

if

1) a is approachable from $\mathrm{dom}(f)$, and

2) For every sequence $\{x_n\}$ in $\mathrm{dom}(f) \setminus \{a\}$

$$\{x_n\} \to a \quad \Longrightarrow \quad \{f(x_n)\} \to L.$$

Note that the value of $f(a)$ (if it exists) has no influence on the meaning of $\lim\limits_{x\to a} f(x) = L$. Also the "$x$" in (10.8) is a dummy variable, and can be replaced by any other symbol that has no assigned meaning.

10.9 Example. For all $a \in \mathbf{R}$ we have

$$\lim_{x\to a} x = a.$$

Also

$$\lim_{x\to a} \cos(x) = \cos(a)$$

and

$$\lim_{x\to a} \sin(x) = \sin(a),$$

by lemma 9.34. Also

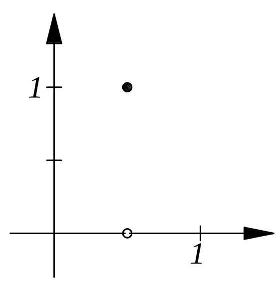
$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

by theorem 9.37.

10.10 Example. $\displaystyle\lim_{x\to 0}\frac{x}{|x|}$ is not defined. Let $x_n=\dfrac{(-1)^n}{n}$. Then $\{x_n\}$ is a sequence in $\mathbf{R}\setminus\{0\}$, and $\{x_n\}\to 0$ and $\dfrac{x_n}{|x_n|}=\dfrac{\frac{(-1)^n}{n}}{(\frac{1}{n})}=(-1)^n$. We know there is no number L such that $\{(-1)^n\}\to L$.

10.11 Example. Let f be the spike function

$$f(x)=\begin{cases}0 & \text{if } x\in\mathbf{R}\setminus\{\frac{1}{2}\}\\ 1 & \text{if } x=\frac{1}{2}.\end{cases}$$



Then $\displaystyle\lim_{x\to\frac{1}{2}} f(x)=0$, since if $\{x_n\}$ is a generic sequence in $\text{dom}(f)\setminus\{\frac{1}{2}\}$, then $\{f(x_n)\}$ is the constant sequence $\{0\}$.

10.12 Example. The limit

$$\lim_{x\to 0}(\sqrt{x}+\sqrt{-x})$$

does not exist. If $f(x)=\sqrt{x}+\sqrt{-x}$, then the domain of f consists of the single point 0, and 0 is not approachable from $\text{dom}(f)$. If we did not have condition **1)** in our definition, we would have

$$\lim_{x\to 0}\sqrt{x}+\sqrt{-x}=0 \text{ and } \lim_{x\to 0}\sqrt{x}+\sqrt{-x}=\pi,$$

which would not be a good thing. (If there are no sequences in $\text{dom}(f) \setminus \{a\}$, then

$$\text{for every sequence } \{x_n\} \text{ in } \text{dom}(f) \setminus \{a\}[\text{statement about } \{x_n\}]$$

is true, no matter what [statement about $\{x_n\}$] is.)

In this course we will not care much about functions like $\sqrt{x} + \sqrt{-x}$.

10.13 Example. I will show that

$$\lim_{x \to a} \sqrt{x} = \sqrt{a} \tag{10.14}$$

for all $a \in \mathbf{R}_{\geq 0}$.

Case 1: Suppose $a \in \mathbf{R}^+$. Let $\{x_n\}$ be a generic sequence in $\mathbf{R}^+ \setminus \{a\}$ such that $\{x_n\} \to a$. Then

$$0 \leq |\sqrt{x}_n - \sqrt{a}| = \left| \frac{\sqrt{x}_n - \sqrt{a}}{1} \cdot \frac{\sqrt{x}_n + \sqrt{a}}{\sqrt{x}_n + \sqrt{a}} \right| = \frac{|x_n - a|}{\sqrt{x}_n + \sqrt{a}} < \frac{|x_n - a|}{\sqrt{a}}.$$

Now, since $\{x_n\} \to a$, we have

$$\lim \left\{ \frac{|x_n - a|}{\sqrt{a}} \right\} = \frac{1}{\sqrt{a}} \lim\{|x_n - a|\} = 0,$$

so by the squeezing rule $\lim\{\sqrt{x_n} - \sqrt{a}\} = 0$ which is equivalent to

$$\lim\{\sqrt{x_n}\} = \sqrt{a}.$$

This proves (10.14) when $a > 0$.

Case 2: Suppose $a = 0$. The domain of the square root function is $[0, \infty)$, and 0 is approachable from this set.

Let $\{x_n\}$ be a sequence in $\mathbf{R}^+$ such that $\{x_n\} \to 0$. To show that $\{\sqrt{x_n}\} \to 0$, I'll use the definition of limit. Let $\epsilon \in \mathbf{R}^+$. Then $\epsilon^2 \in \mathbf{R}^+$, so by the definition of convergence, there is an $N(\epsilon^2) \in \mathbf{Z}^+$ such that for all $n \in \mathbf{Z}_{\geq N(\epsilon^2)}$ we have $(x_n = |x_n - 0| < \epsilon^2)$. Then for all $n \in \mathbf{Z}_{\geq N(\epsilon^2)}$ we have $(\sqrt{x_n} = |\sqrt{x_n} - 0| < \epsilon)$ and hence $\{\sqrt{x}_n\} \to 0$. |||

Many of our rules for limits of sequences have immediate corollaries as rules for limits of functions. For example, suppose f, g are real valued functions with

$\text{dom}(f) \subset \mathbf{R}$ and $\text{dom}(g) \subset \mathbf{R}$. Suppose $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$. Let $\{x_n\}$ be a generic sequence in $\Big(\text{dom}(f) \cap \text{dom}(g)\Big) \setminus \{a\}$ such that $\{x_n\} \to a$. Then $\{x_n\}$ is a sequence in $\text{dom}(f) \setminus \{a\}$ and $\{x_n\} \to a$, so

$$\{f(x_n)\} \to L.$$

Also $\{x_n\}$ is a sequence in $\text{dom}(g) \setminus \{a\}$ and $\{x_n\} \to a$ so

$$\{g(x_n)\} \to M.$$

By the sum and product rules for sequences, for any $c \in \mathbf{R}$

$$\begin{aligned} \{(f \pm g)(x_n)\} &= \{f(x_n) \pm g(x_n)\} \to L \pm M, \\ \{(fg)(x_n)\} &= \{f(x_n)g(x_n)\} \to LM, \end{aligned}$$

and

$$\{(cf)(x_n)\} = \{c \cdot f(x_n)\} \to cL,$$

and thus we've proved that

$$\begin{aligned} \lim_{x\to a}(f \pm g)(x) &= L \pm M = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \\ \lim_{x\to a}(fg)(x) &= LM = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \end{aligned}$$

and

$$\lim_{x\to a}(cf)(x) = cL = c \lim_{x\to a} f(x).$$

Moreover if $a \in \text{dom}(\frac{f}{g})$ (so that $g(a) \neq 0$), and if $x_n \in \text{dom}(\frac{f}{g})$ for all x_n (so that $g(x_n) \neq 0$ for all n), it follows from the quotient rule for sequences that

$$\Big\{(\frac{f}{g})(x_n)\Big\} = \Big\{\frac{f(x_n)}{g(x_n)}\Big\} \to \frac{L}{M},$$

so that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} \text{ if } \lim_{x\to a} g(x) \neq 0.$$

Actually all of the results just claimed are not quite true as stated. For we have

$$\lim_{x\to 0} \sqrt{x} = 0$$

and

$$\lim_{x\to 0} \sqrt{-x} = 0$$

but

$$\lim_{x\to 0} \sqrt{x} + \sqrt{-x} \text{ does not exist!}$$

The correct theorem is:

10.15 Theorem (Sum, product, quotient rules for limits.) *Let* f, g *be real valued functions with* $\text{dom}(f) \subset \mathbf{R}$ *and* $\text{dom}(g) \subset \mathbf{R}$*, and let* $c \in \mathbf{R}$*. Suppose* $\lim_{x\to a} f(x)$*, and* $\lim_{x\to a} g(x)$ *both exist. Then* if a is approachable from $\text{dom}(f) \cap \text{dom}(g)$ *we have*

$$\begin{aligned} \lim_{x\to a}(f \pm g)(x) &= \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \\ \lim_{x\to a}(fg)(x) &= \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) \\ \lim_{x\to a}(cf)(x) &= c \cdot \lim_{x\to a} f(x). \end{aligned}$$

If in addition $\lim_{x\to a} g(x) \neq 0$ *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}.$$

Proof: Most of the theorem follows from the remarks made above. We will assume the remaining parts.

10.16 Theorem (Inequality rule for limits of functions.) *Let* f *and* g *be real functions with* $\text{dom}(f) \subset \mathbf{R}$ *and* $\text{dom}(g) \subset \mathbf{R}$*. Suppose that*

i $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *both exist.*

ii a *is approachable from* $\text{dom}(f) \cap \text{dom}(g)$*.*

iii *There is a positive number* ϵ *such that*

$$f(x) \le g(x) \textit{ for all } x \textit{ in } \text{dom}(f) \cap \text{dom}(g) \cap (a - \epsilon, a + \epsilon).$$

Then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$*.*

Proof: Let $\{x_n\}$ be a sequence in $\Big(\text{dom}(f)\cap\text{dom}(g)\cap(a-\epsilon,a+\epsilon)\Big)\setminus\{a\}$ such that $\{x_n\}\to a$. Then $\{x_n\}$ is a sequence in $\text{dom}(f)\setminus\{a\}$ that converges to a, so by the definition of limit of a function,

$$\lim\{f(x_n)\} = \lim_{x\to a} f(x).$$

Similiarly

$$\lim\{g(x_n)\} = \lim_{x\to a} g(x).$$

Also $f(x_n)\leq g(x_n)$ for all n, so it follows from the inequality rule for limits of sequences that $\lim\{f(x_n)\}\leq\lim\{g(x_n)\}$, i.e. $\lim_{x\to a} f(x)\leq\lim_{x\to a} g(x)$. |||.

10.17 Theorem (Squeezing rule for limits of functions.) *Let f, g and h be real functions with* $\text{dom}(f)\subset\mathbf{R}$, $\text{dom}(g)\subset\mathbf{R}$, *and* $\text{dom}(h)\subset\mathbf{R}$. *Suppose that*

- **i** $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} h(x)$ *both exist and are equal.*
- **ii** a *is approachable from* $\text{dom}(f)\cap\text{dom}(g)\cap\text{dom}(h)$.
- **iii** *There is a positive number ϵ such that $f(x)\leq g(x)\leq h(x)$ for all x in* $\text{dom}(f)\cap\text{dom}(g)\cap\text{dom}(h)\cap(a-\epsilon,a+\epsilon)$.

Then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \lim_{x\to a} h(x)$.

Proof: The proof is almost identical to the proof of theorem 10.16.

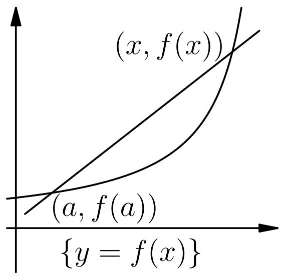
10.3 Definition of the Derivative.

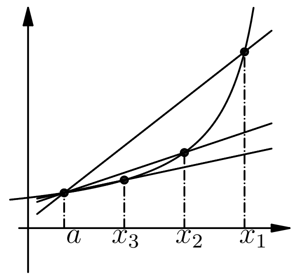
Our definition of tangent to a curve will be based on the following definition:

10.18 Definition (Derivative.) Let f be a real valued function such that $\text{dom}(f)\subset\mathbf{R}$. Let $a\in\text{dom}(f)$. We say that f is *differentiable at a* if a is an interior point of $\text{dom}(f)$ and the limit

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \tag{10.19}$$

exists. In this case we denote the limit in (10.19) by $f'(a)$, and we call $f'(a)$ the *derivative of f at a*.





The quantity $\dfrac{f(x)-f(a)}{x-a}$ represents the slope of the line joining the points $\big(a, f(a)\big)$ and $\big(x, f(x)\big)$ on the graph of f. If x and a are different points in dom(f) then this quotient will be defined. If we choose a sequence of points $\{x_n\}$ converging to a, and if the slopes $\left\{\dfrac{f(x_n)-f(a)}{x_n-a}\right\}$ converge to a number m which is independent of the sequence $\{x_n\}$, then it is reasonable to call m (i.e., $f'(a)$) the *slope of the tangent line to the graph of f at* $\big(a, f(a)\big)$.

10.20 Definition (Tangent to the graph of a function.) Let f be a real valued function with dom$(f) \subset \mathbf{R}$, and let $a \in$ dom(f). If f is differentiable at a then we define the *slope of the tangent to* graph(f) at the point $\big(a, f(a)\big)$ to be the number $f'(a)$, and we define the *tangent to* graph(f) at $\big(a, f(a)\big)$ to be the line that passes through $\big(a, f(a)\big)$ with slope $f'(a)$.

Remark: This definition will need to be generalized later to apply to curves that are not graphs of functions. Also this definition does not allow vertical lines to be tangents, whereas on geometrical grounds, vertical tangents are quite reasonable.

10.21 Example. We will calculate the tangent to $\{y = x^3\}$ at a generic point (a, a^3).

Let $f(x) = x^3$. Then for all $a \in \mathbf{R}$,

$$\begin{aligned}\lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{x^3-a^3}{x-a} = \lim_{x\to a}\frac{(x-a)(x^2+ax+a^2)}{(x-a)} \\ &= \lim_{x\to a}(x^2+ax+a^2) = a^2+a^2+a^2 = 3a^2.\end{aligned}$$

Hence the tangent line to graph(f) at (a, a^3) is the line through (a, a^3) with slope $3a^2$, and the equation of the tangent line is

$$y - a^3 = 3a^2(x - a)$$

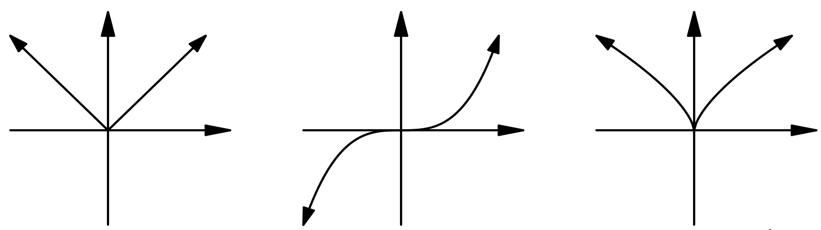
or

$$y = a^3 + 3a^2x - 3a^3 = 3a^2x - 2a^3$$

or

$$y = a^2(3x - 2a).$$

10.22 Example. We will now consider some of the examples on page 220.



If $f(x) = |x|$ then

$$\frac{f(x) - f(0)}{x - 0} = \frac{|x|}{x}.$$

We saw in example 10.10 that $\lim\limits_{x \to 0} \frac{|x|}{x}$ does not exist. Hence, the graph of f at $(0, 0)$ has no tangent.

If $g(x) = x^3$, then in the previous example we saw that the equation of the tangent to graph(g) at $(0, 0)$ is $y = 0$; i.e., the x-axis is tangent to the curve. Note that in this case the tangent line crosses the curve at the point of tangency.

If $h(x) = x$ then for all $a \in \mathbf{R}$,

$$\lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = \lim_{x \to a} 1 = 1.$$

The equation of the tangent line to graph(h) at (a, a) is

$$y = a + 1(x - a)$$

or $y = x$.

Thus at each point on the curve the tangent line coincides with the curve.

Let $k(x) = (x^2)^{1/3}$. This is not the same as the function $l(x) = x^{2/3}$ since the domain of l is $\mathbf{R}_{\geq 0}$ while the domain of k is $\mathbf{R}$. (For all $x \in \mathbf{R}$ we have $x^2 \in \mathbf{R}_{\geq 0} = \mathrm{dom}(g)$ where $g(x) = x^{1/3}$.)

I want to investigate $\displaystyle\lim_{x\to 0} \frac{k(x) - k(0)}{x - 0} = \lim_{x\to 0} \frac{k(x)}{x}$. From the picture, I expect this graph to have an infinite slope at $(0, 0)$, which means according to our definition that there is no tangent line at $(0, 0)$. Let $\{x_n\} = \left\{\dfrac{1}{n^3}\right\}$. Then $\{x_n\} \to 0$, but

$$\frac{k(x_n)}{x_n} = \frac{\left(\frac{1}{n^6}\right)^{1/3}}{\left(\frac{1}{n^3}\right)} = \frac{\frac{1}{n^2}}{\frac{1}{n^3}} = n$$

so $\lim \left\{\dfrac{k(x_n)}{x_n}\right\}$ does not exist and hence $\displaystyle\lim_{x\to 0} \frac{k(x)}{x}$ does not exist.

10.23 Example. Let $f(x) = \sqrt{x}$ for $x \in \mathbf{R}_{\geq 0}$. Let $a \in \mathbf{R}^+$ and let $x \in \mathrm{dom}(f) \setminus \{a\}$. Then

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{\sqrt{x} - \sqrt{a}}{(\sqrt{x})^2 - (\sqrt{a})^2} \\ &= \frac{\sqrt{x} - \sqrt{a}}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}. \end{aligned}$$

Hence

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}}; \qquad (10.24)$$

i.e.,

$$f'(a) = \frac{1}{2\sqrt{a}} \text{ for all } a \in \mathbf{R}^+.$$

In line (10.24) I used the fact that $\displaystyle\lim_{x\to a} \sqrt{x} = \sqrt{a}$, together with the sum and quotient rules for limits.

10.25 Exercise. Let $f(x) = \frac{1}{x}$. Sketch the graph of f. For what values of x do you expect $f'(x)$ to be -1? For what values of x do you expect $f'(x)$ to be positive? What do you expect to happen to $f'(x)$ when x is a small positive number? What do you expect to happen to $f'(x)$ when x is a small negative number?

Calculate $f'(a)$ for arbitrary $a \in \text{dom}(f)$. Does your answer agree with your prediction?

10.26 Exercise. Let $f(x) = \sin(x)$ for $-\pi < x < 4\pi$. Sketch the graph of f. Use the same scale on the x-axis and the y-axis.

On what intervals do you expect $f'(x)$ to be positive? On what intervals do you expect $f'(x)$ to be negative? Calculate $f'(0)$.

On the basis of symmetry, what do you expect to be the values of $f'(\pi)$, $f'(2\pi)$ and $f'(3\pi)$? For what x do you expect $f'(x)$ to be zero? On the basis of your guesses and your calculated value of $f'(0)$, draw a graph of f', where f' is the function that assigns $f'(x)$ to a generic number x in $(-\pi, 4\pi)$. On the basis of your graph, guess a formula for $f'(x)$.
(Optional) Prove that your guess is correct. (Some trigonometric identities will be needed.)

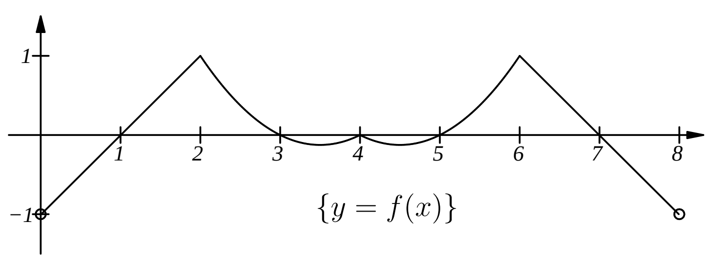
10.27 Exercise. Calculate $f'(x)$ if $f(x) = \frac{x}{x+1}$.

10.28 Exercise.

a) Find $f'(x)$ if $f(x) = x^2 - 2x$.

b) Find the equations for all the tangent lines to graph(f) that pass through the point $(0, -4)$. Make a sketch of graph(f) and the tangent lines.

10.29 Exercise. Consider the function $f\colon (0, 8) \to \mathbf{R}$ whose graph is shown below.



For what x in $(0, 8)$ does $f'(x)$ exist? Sketch the graphs of f and f' on the same set of axes.

The following definition which involves time and motion and particles is not a part of our official development and will not be used for proving any theorems.

10.30 Definition (Velocity.) Let a particle $\mathbf{p}$ move on a number line in such a way that its coordinate at time t is $x(t)$, for all t in some interval J. (Here time is thought of as being specified by a number.) If t_0, t_1 are points in J with $t_0 < t_1$, then the *average velocity* of $\mathbf{p}$ for the time interval $[t_0, t_1]$ is defined to be

$$\frac{x(t_1) - x(t_0)}{t_1 - t_0} = \frac{\text{change in position}}{\text{change in time}}.$$

Note that $x(t_1) - x(t_0)$ is not necessarily the same as the distance moved in the time interval $[t_0, t_1]$. For example, if $x(t) = t(1 - t)$ then $x(1) - x(0) = 0$, but the distance moved by $\mathbf{p}$ in the time interval $[0, 1]$ is $\frac{1}{2}$. (The particle moves from 0 to $\frac{1}{4}$ at time $t = \frac{1}{2}$, and then back to 0.)

The *instantaneous velocity* of $\mathbf{p}$ at a time $t_0 \in J$ is defined to be

$$\lim_{t \to t_0} \frac{x(t) - x(t_0)}{t - t_0} = x'(t_0)$$

provided this limit exists. (If the limit does not exist, then the instantaneous velocity of $\mathbf{p}$ at t_0 is not defined.) If we draw the graph of the function x; i.e., $\{\big(t, x(t)\big) : t \in J\}$, then the velocity of $\mathbf{p}$ at time t_0 is by definition $x'(t_0) =$ slope of tangent to graph(x) at $\big(t_0, x(t_0)\big)$.

In applications we will usually express velocity in units like $\frac{\text{miles}}{\text{hour}}$. We will wait until we have developed some techniques for differentiation before we do any velocity problems.

The definition of velocity just given would have made no sense to Euclid or Aristotle. The Greek theory of proportion does not allow one to divide a length by a time, and Aristotle would no more divide a length by a time than he would add them. Question: Why is it that today in physics you are allowed to divide a length by a time, but you are not allowed to add a length to a time?

In Newton's calculus, the notion of instantaneous velocity or *fluxion* was taken as an undefined, intuitively understood concept, and the fluxions were calculated using methods similar to that used in the example on page 222.

The first "rigorous" definitions of limit of a function were given around 1820 by Bernard Bolzano (1781-1848) and Augustin Cauchy (1789-1857)[23, chapter 1]. The definition of limit of a function in terms of limits of sequences was given by Eduard Heine in 1872.