Chapter 8

Integrable Functions

8.1 Definition of the Integral

If f is a monotonic function from an interval $[a,b]$ to $\mathbf{R}_{\geq 0}$, then we have shown that for every sequence $\{P_n\}$ of partitions on $[a,b]$ such that $\{\mu(P_n)\} \to 0$, and every sequence $\{S_n\}$ such that for all $n \in \mathbf{Z}^+$ S_n is a sample for P_n, we have

$$\{\sum(f, P_n, S_n)\} \to A_a^b f.$$

8.1 Definition (Integral.) Let f be a bounded function from an interval $[a,b]$ to $\mathbf{R}$. We say that f is *integrable on* $[a,b]$ if there is a number V such that for every sequence of partitions $\{P_n\}$ on $[a,b]$ such that $\{\mu(P_n)\} \to 0$, and every sequence $\{S_n\}$ where S_n is a sample for P_n

$$\{\sum(f, P_n, S_n)\} \to V.$$

If f is integrable on $[a,b]$ then the number V just described is denoted by $\displaystyle\int_a^b f$ and is called "the integral from a to b of f." Notice that by our definition an integrable function is necessarily bounded.

The definition just given is essentially due to Bernhard Riemann(1826–1866), and first appeared around 1860[39, pages 239 ff]. The symbol $\displaystyle\int$ was introduced by Leibniz sometime around 1675[15, vol 2, p242]. The symbol is a form of the letter s, standing for *sum* (in Latin as well as in English.) The practice of attaching the limits a and b to the integral sign was introduced by

Joseph Fourier around 1820. Before this time the limits were usually indicated by words.

We can now restate theorems 7.6 and 7.15 as follows:

8.2 Theorem (Monotonic functions are integrable I.) *If f is a monotonic function on an interval $[a, b]$ with non-negative values, then f is integrable on $[a, b]$ and*

$$\int_a^b f = A_a^b f = \alpha(\{(x, y) \colon a \le x \le b \text{ and } 0 \le y \le f(x)\}).$$

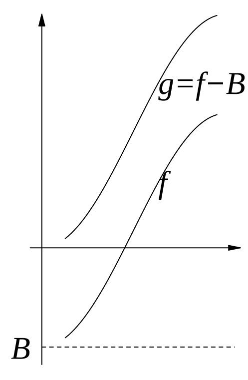
8.3 Theorem (Integrals of power functions.) *Let $r \in \mathbf{Q}$, and let a, b be real numbers such that $0 < a \le b$. Let $f_r(x) = x^r$ for $a \le x \le b$. Then*

$$\int_a^b f_r = \begin{cases} \dfrac{b^{r+1} - a^{r+1}}{r+1} & \text{if } r \in \mathbf{Q} \setminus \{-1\} \\ \ln(b) - \ln(a) & \text{if } r = -1. \end{cases}$$

In general integrable functions may take negative as well as positive values and in these cases $\displaystyle\int_a^b f$ does not represent an area.

The next theorem shows that monotonic functions are integrable even if they take on negative values.

8.4 Example (Monotonic functions are integrable II.) Let f be a monotonic function from an interval $[a, b]$ to $\mathbf{R}$. Let B be a non-positive number such that $f(x) \ge B$ for all $x \in [a, b]$. Let $g(x) = f(x) - B$.



Then g is a monotonic function from $[a,b]$ to $\mathbf{R}_{\geq 0}$. Hence by theorem 7.6, g is integrable on $[a,b]$ and $\int_a^b g = A_a^b(g)$. Now let $\{P_n\}$ be a sequence of partitions of $[a,b]$ such that $\{\mu(P_n)\} \to 0$, and let $\{S_n\}$ be a sequence such that for each n in $\mathbf{Z}^+$, S_n is a sample for P_n. Then

$$\{\sum(g, P_n, S_n)\} \to A_a^b(g). \tag{8.5}$$

If $P_n = \{x_0, \cdots, x_m\}$ and $S_n = \{s_1, \cdots, s_m\}$ then

$$\begin{aligned}
\sum(g, P_n, S_n) &= \sum_{i=1}^m g(s_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^m \Big(f(s_i) - B\Big)(x_i - x_{i-1}) \\
&= \sum_{i=1}^m f(s_i)(x_i - x_{i-1}) - B\sum_{i=1}^m (x_i - x_{i-1}) \\
&= \sum(f, P_n, S_n) - B(b-a).
\end{aligned}$$

Thus by (8.5)

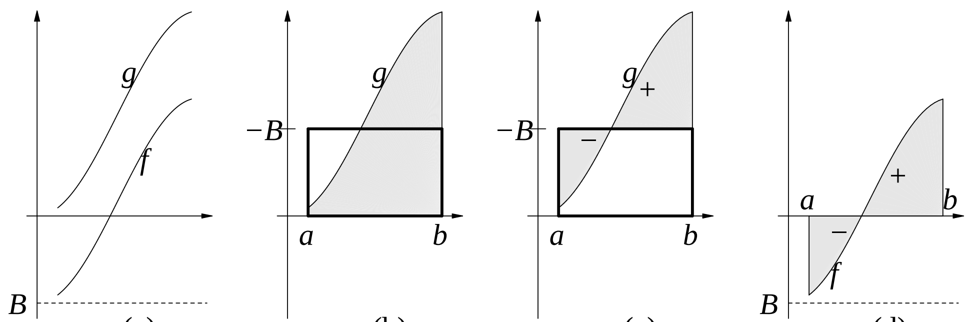
$$\{\sum(f, P_n, S_n) - B(b-a)\} \to A_a^b(g).$$

If we use the fact that $\{B(b-a)\} \to B(b-a)$, and then use the sum theorem for limits of sequences, we get

$$\{\sum(f, P_n, S_n)\} \to A_a^b(g) + B(b-a).$$

It follows from the definition of integrable functions that f is integrable on $[a,b]$ and

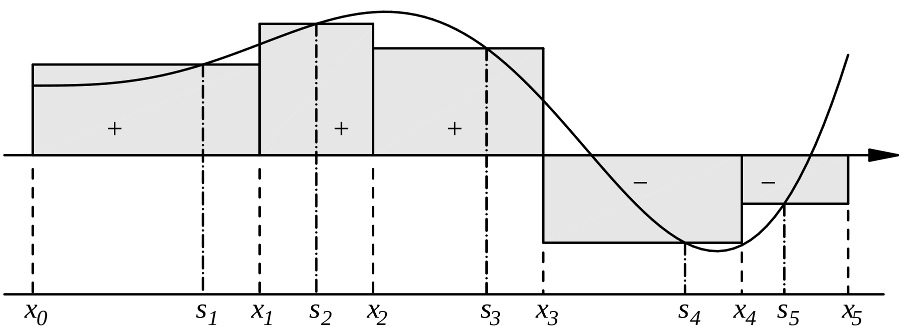
$$\int_a^b f = A_a^b(g) + B(b-a) = \int_a^b g + B(b-a) = \int_a^b g - |B|(b-a).$$



(a) (b) (c) (d)

Thus in figure b, $\displaystyle\int_a^b f$ represents the shaded area with the area of the thick box subtracted from it, which is the same as the area of the region marked "+" in figures c and d, with the area of the region marked "−" subtracted from it.

The figure represents a geometric interpretation for a Riemann sum. In the figure



$$f(s_i) > 0 \text{ for } i = 1, 2, 3, \quad f(s_i) < 0 \text{ for } i = 4, 5.$$

$$\sum_{i=1}^{3} f(s_i)(x_i - x_{i-1})$$

is the area of $\displaystyle\bigcup_{i=1}^{3} B\Big(x_{i-1}, x_i\colon 0, f(s_i)\Big)$ and

$$\sum_{i=4}^{5} f(s_i)(x_i - x_{i-1})$$

is the *negative* of the area of

$$\bigcup_{i=4}^{5} B(x_{i-1}, x_i : f(s_i), 0).$$

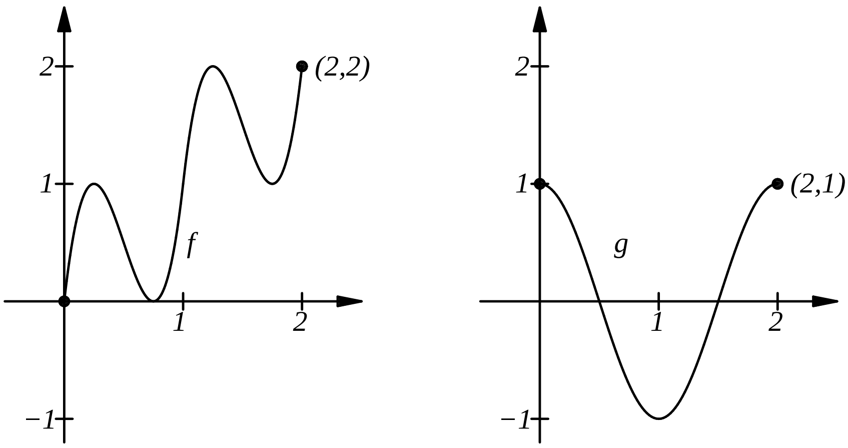
In general you should think of $\displaystyle\int_a^b f$ as the difference $\alpha(S^+) - \alpha(S^-)$ where

$$S^+ = \{(x, y)\colon a \le x \le b \text{ and } 0 \le y \le f(x)\}$$

and

$$S^- = \{(x, y) \colon a \le x \le b \text{ and } f(x) \le y \le 0\}.$$

8.6 Exercise. The graphs of two functions f, g from $[0, 2]$ to $\mathbf{R}$ are sketched below.



Let

$$F(x) = \big(f(x)\big)^2 \text{ for } 0 \le x \le 2, \quad G(x) = \big(g(x)\big)^2 \text{ for } 0 \le x \le 2.$$

Which is larger:

a) $\displaystyle\int_0^1 f$ or $\displaystyle\int_0^1 F$?

b) $\displaystyle\int_0^1 g$ or $\displaystyle\int_0^1 G$?

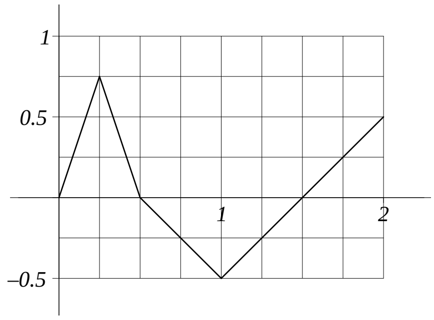
c) $\displaystyle\int_0^1 f$ or $\displaystyle\int_0^1 g$?

d) $\displaystyle\int_0^{1/2} g$ or $\displaystyle\int_0^{1/2} G$?

e) $\displaystyle\int_0^2 g$ or $\displaystyle\int_0^2 G$?

Explain how you decided on your answers. Your explanations may be informal, but they should be convincing.

8.7 Exercise. Below is the graph of a function g. By looking at the graph of g estimate the following integrals. (No explanation is necessary.)



Graph of g

a) $\int_{\frac{1}{4}}^{\frac{3}{4}} g$.

b) $\int_{1}^{2} g$.

c) $\int_{0}^{\frac{3}{4}} g$.

8.8 Exercise. Sketch the graph of one function f satisfying all four of the following conditions.

a) $\int_{0}^{1} f = 1$.

b) $\int_{0}^{2} f = -1$.

c) $\int_{0}^{3} f = 0$.

d) $\int_{0}^{4} f = 1$.

8.2 Properties of the Integral

8.9 Definition (Operations on functions.) Let $f: S \to \mathbf{R}$ and $g: T \to \mathbf{R}$ be functions where S, T are sets. Let $c \in \mathbf{R}$. We define functions $f \pm g$, fg, cf, $\frac{f}{g}$ and $|f|$ as follows:

$$\begin{array}{l}
(f+g)(x) = f(x) + g(x) \text{ for all } x \in S \cap T. \\
(f-g)(x) = f(x) - g(x) \text{ for all } x \in S \cap T. \\
(fg)(x) = f(x)g(x) \text{ for all } x \in S \cap T. \\
(cf)(x) = c \cdot f(x) \text{ for all } x \in S. \\
\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \text{ for all } x \in S \cap T \text{ such that } g(x) \neq 0. \\
|f|(x) = |f(x)| \text{ for all } x \in S.
\end{array}$$

Remark: These operations of addition, subtraction, multiplication and division for functions satisfy the associative, commutative and distributive laws that you expect them to. The proofs are straightforward and will be omitted.

8.10 Definition (Partition-sample sequence.) Let $[a, b]$ be an interval. By a *partition-sample sequence* for $[a, b]$ I will mean a pair of sequences $(\{P_n\}, \{S_n\})$ where $\{P_n\}$ is a sequence of partitions of $[a, b]$ such that $\{\mu(P_n)\} \to 0$, and for each n in $\mathbf{Z}^+$, S_n is a sample for P_n.

8.11 Theorem (Sum theorem for integrable functions.) *Let f, g be integrable functions on an interval $[a, b]$. Then $f \pm g$ and cf are integrable on $[a, b]$ and*

$$\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g,$$

and

$$\int_a^b cf = c \int_a^b f.$$

Proof: Suppose f and g are integrable on $[a, b]$. Let $(\{P_n\}, \{S_n\})$ be a partition-sample sequence for $[a, b]$. If $P_n = \{x_0, \cdots, x_m\}$ and $S_n = \{s_1, \cdots, s_m\}$, then

$$\sum (f \pm g, P_n, S_n) \quad = \quad \sum_{i=1}^{m} (f \pm g)(s_i)(x_i - x_{i-1})$$

$$\begin{aligned}
&= \sum_{i=1}^{m} \Big(f(s_i) \pm g(s_i)\Big)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{m} f(s_i)(x_i - x_{i-1}) \pm \sum_{i=1}^{m} g(s_i)(x_i - x_{i-1}) \\
&= \sum(f, P_n, S_n) \pm \sum(g, P_n, S_n).
\end{aligned}$$

Since f and g are integrable, we have

$$\{\sum(f, P_n, S_n)\} \to \int_a^b f \text{ and } \{\sum(g, P_n, S_n)\} \to \int_a^b g.$$

By the sum theorem for sequences,

$$\{\sum(f \pm g), P_n, S_n)\} = \{\sum(f, P_n, S_n) \pm \sum(g, P_n, S_n)\} \to \int_a^b f \pm \int_a^b g.$$

Hence $f \pm g$ is integrable and $\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$. The proof of the second statement is left as an exercise.

8.12 Notation ($\int_a^b f(t)\, dt$) If f is integrable on an interval $[a, b]$ we will sometimes write $\int_a^b f(x)\, dx$ instead of $\int_a^b f$. The "x" in this expression is a dummy variable, but the "d" is a part of the notation and may not be replaced by another symbol. This notation will be used mainly in cases where no particular name is available for f. Thus

$$\int_1^2 t^3 + 3t\, dt \text{ or } \int_1^2 x^3 + 3x\, dx \text{ or } \int_1^2 (x^3 + 3x) dx$$

means $\int_1^2 F$ where F is the function on $[1, 2]$ defined by $F(t) = t^3 + 3t$ for all $t \in [1, 2]$. The "d" here stands for difference, and dx is a ghost of the differences $x_i - x_{i-1}$ that appear in the approximations for the integral. The dx notation is due to Leibniz.

8.13 Example. Let

$$f(x) = (x-1)^2 - \frac{1}{x} + \frac{3}{\sqrt{x}} = x^2 - 2x + x^0 - \frac{1}{x} + 3x^{-\frac{1}{2}}.$$

This function is integrable over every closed bounded subinterval of $(0, \infty)$, since it is a sum of five functions that are known to be integrable. By several applications of the sum theorem for integrals we get

$$\begin{aligned}\int_1^2 f &= \int_1^2 (x^2 - 2x + 1 - \frac{1}{x} + 3x^{-\frac{1}{2}})dx \\ &= \left(\frac{2^3 - 1^3}{3}\right) - 2\left(\frac{2^2 - 1^2}{2}\right) + \left(\frac{2^1 - 1^1}{1}\right) - \ln(2) + 3\left(\frac{2^{\frac{1}{2}} - 1^{\frac{1}{2}}}{\frac{1}{2}}\right) \\ &= \frac{7}{3} - 3 + 1 - \ln(2) + 6(\sqrt{2} - 1) = -\frac{17}{3} - \ln(2) + 6\sqrt{2}.\end{aligned}$$

8.14 Exercise. Calculate the following integrals.

a) $\displaystyle\int_1^a (2 - x)^2 dx$. Here $a > 1$.

b) $\displaystyle\int_1^4 \sqrt{x} - \frac{1}{x^2}\, dx$.

c) $\displaystyle\int_1^{27} x^{-\frac{1}{3}} dx$.

d) $\displaystyle\int_0^{27} x^{-\frac{1}{3}} dx$.

e) $\displaystyle\int_1^2 \frac{x+1}{x} dx$.

f) $\displaystyle\int_a^b M\, dx$. Here $a \leq b$, and M denotes a constant function.

8.15 Theorem (Inequality theorem for integrals.) *Let f and g be integrable functions on the interval $[a, b]$ such that*

$$f(x) \leq g(x) \text{ for all } x \in [a, b].$$

Then

$$\int_a^b f \leq \int_a^b g.$$

8.16 Exercise. Prove the inequality theorem for integrals.

8.17 Corollary. *Let f be an integrable function on the interval $[a,b]$. Suppose $|f(x)| \leq M$ for all $x \in [a,b]$. Then*

$$\left|\int_a^b f\right| \leq M(b-a).$$

Proof: We have

$$-M \leq f(x) \leq M \text{ for all } x \in [a,b].$$

Hence by the inequality theorem for integrals

$$\int_a^b -M \leq \int_a^b f \leq \int_a^b M.$$

Hence

$$-M(b-a) \leq \int_a^b f \leq M(b-a).$$

It follows that

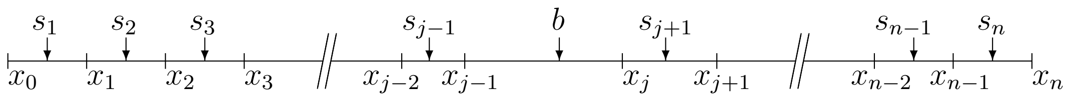
$$\left|\int_a^b f\right| \leq M(b-a). \parallel$$

8.18 Theorem. *Let a,b,c be real numbers with $a < b < c$, and let f be a function from $[a,c]$ to $\mathbf{R}$. Suppose f is integrable on $[a,b]$ and f is integrable on $[b,c]$. Then f is integrable on $[a,c]$ and $\displaystyle\int_a^c f = \int_a^b f + \int_b^c f$.*

Proof: Since f is integrable on $[a,b]$ and on $[b,c]$, it follows that f is bounded on $[a,b]$ and on $[b,c]$, and hence f is bounded on $[a,c]$. Let $(\{P_n\},\{S_n\})$ be a partition-sample sequence for $[a,c]$. For each n in $\mathbf{Z}^+$ we define a partition P_n' of $[a,b]$ and a partition P_n'' of $[b,c]$, and a sample S_n' for P_n', and a sample S_n'' for P_n'' as follows:

$$\text{Let } P_n = \{x_0, x_1, \cdots, x_m\}, \;\; S_n = \{s_1, s_2, \cdots, s_m\}.$$

Then there is an index j such that $x_{j-1} \leq b \leq x_j$.



Let

$$\begin{aligned} P'_n &= \{x_0, \cdots, x_{j-1}, b\}, \quad P''_n = \{b, x_j, \cdots, x_m\} && (8.19) \\ S'_n &= \{s_1, \cdots, s_{j-1}, b\}, \quad S''_n = \{b, s_{j+1}, \cdots, s_m\} && (8.20) \end{aligned}$$

We have

$$\begin{aligned} \sum(f, P'_n, S'_n) &+ \sum(f, P''_n, S''_n) \\ &= \sum_{i=1}^{j-1} f(s_i)(x_i - x_{i-1}) + f(b)(b - x_{j-1}) + f(b)(x_j - b) \\ &\quad + \sum_{i=j+1}^{m} f(s_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{m} f(s_i)(x_i - x_{i-1}) + f(b)(x_j - x_{j-1}) - f(s_j)(x_j - x_{j-1}) \\ &= \sum(f, P_n, S_n) + \Delta_n, && (8.21) \end{aligned}$$

where

$$\Delta_n = \Big(f(b) - f(s_j)\Big)(x_j - x_{j-1}).$$

Let M be a bound for f on $[a, c]$. Then

$$|f(b) - f(s_j)| \le |f(b)| + |f(s_j)| \le M + M = 2M.$$

Also,

$$(x_j - x_{j-1}) \le \mu(P_n).$$

Now

$$0 \le |\Delta_n| = |f(b) - f(s_j)| \cdot |x_j - x_{j-1}| \le 2M\mu(P_n).$$

Since

$$\lim\{2M\mu(P_n)\} = 0,$$

it follows from the squeezing rule that $\{|\Delta_n|\} \to 0$ and hence $\{\Delta_n\} \to 0$.

From equation (8.21) we have

$$\sum(f, P_n, S_n) = \sum(f, P'_n, S'_n) + \sum(f, P''_n, S''_n) - \Delta_n. \tag{8.22}$$

Since $\mu(P'_n) \le \mu(P_n)$ and $\mu(P''_n) \le \mu(P_n)$, we see that $(\{P'_n\}, \{S'_n\})$ is a partition-sample sequence on $[a, b]$, and $(\{P''_n\}, \{S''_n\})$ is a partition-sample sequence on $[b,c]$. Since f was given to be integrable on $[a,b]$ and on $[b,c]$, we know that

$$\{\sum(f, P_n', S_n')\} \to \int_a^b f$$

and

$$\{\sum(f, P_n'', S_n'')\} \to \int_b^c f.$$

Hence it follows from (8.22) that

$$\{\sum(f, P_n, S_n)\} \to \int_a^b f + \int_b^c f$$

i.e., f is integrable on $[a,c]$ and

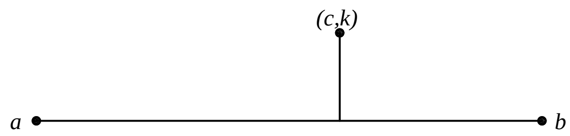
$$\int_a^c f = \int_a^b f + \int_b^c f. \parallel$$

8.23 Corollary. *Let $a_1, a_2, \cdots, a_n$ be real numbers with $a_1 \leq a_2 \cdots \leq a_n$, and let f be a bounded function on $[a_1, a_n]$. If the restriction of f to each of the intervals $[a_1,a_2], [a_2,a_3], \cdots, [a_{n-1}, a_n]$ is integrable, then f is integrable on $[a_1, a_n]$ and*

$$\int_{a_1}^{a_n} f = \int_{a_1}^{a_2} f + \int_{a_2}^{a_3} f + \cdots + \int_{a_{n-1}}^{a_n} f.$$

8.24 Definition (Spike function.) Let $[a,b]$ be an interval. A function $f : [a,b] \to \mathbf{R}$ is called a *spike function*, if there exist numbers c and k, with $c \in [a,b]$ such that

$$f(x) = \begin{cases} 0 & \text{if } x \in [a,b] \setminus \{c\} \\ k & \text{if } x = c. \end{cases}$$



area under graph of spike function

8.25 Theorem (Spike functions are integrable.) *Let a, b, c, k be real numbers with $a < c$ and $a \leq b \leq c$. Let*

$$f(x) = \begin{cases} 0 & \text{if } x \in [a, c] \setminus \{b\} \\ k & \text{if } x = b. \end{cases}$$

Then f is integrable on $[a, c]$ and $\int_a^c f = 0$.

Proof: Case 1: Suppose $k \geq 0$. Observe that f is increasing on the interval $[a, b]$ and decreasing on the interval $[b, c]$, so f is integrable on each of these intervals. The set of points under the graph of f is the union of a horizontal segment and a vertical segment, and thus is a zero-area set. Hence

$$\int_a^b f = A_a^b f = 0 \qquad \int_b^c f = A_b^c f = 0.$$

By the previous theorem, f is integrable on $[a, c]$, and

$$\int_a^c f = \int_a^b f + \int_b^c f = 0 + 0 = 0$$

.

Case 2: Suppose $k < 0$. Then by case 1 we see that $-f$ is integrable with integral equal to zero, so by the sum theorem for integrals $\int f = 0$ too. |||

8.26 Corollary. *Let a, b, c, k be real numbers with $a < c$ and $a \leq b \leq c$. Let $f \colon [a, c] \to \mathbf{R}$ be an integrable function and let $g \colon [a, c] \to \mathbf{R}$ be defined by*

$$g(x) = \begin{cases} f(x) & \text{if } x \in [a, c] \setminus \{b\} \\ k & \text{if } x = b. \end{cases}$$

Then g is integrable on $[a, c]$ and $\int_a^c g = \int_a^c f$.

8.27 Corollary. *Let f be an integrable function from an interval $[a, b]$ to $\mathbf{R}$. Let $a_1 \cdots a_n$ be a finite set of distinct points in $\mathbf{R}$, and let $k_1 \cdots k_n$ be a finite set of numbers. Let*

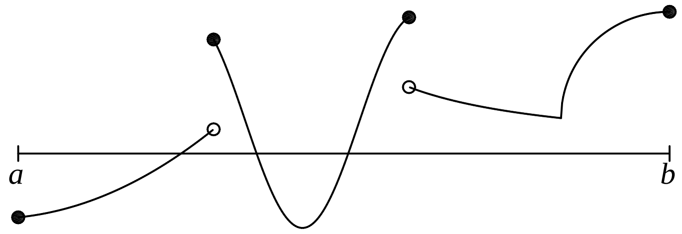
$$g(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \setminus \{a_1, \cdots, a_n\} \\ k_j & \text{if } x = a_j \text{ for some } j \text{ with } 1 \leq j \leq n. \end{cases}$$

Then g is integrable on $[a, b]$ and $\int_a^b f = \int_a^b g$. Thus we can alter an integrable function on any finite set of points without changing its integrability or its integral.

8.28 Exercise. Prove corollary 8.26, i.e., explain why it follows from theorem 8.25.

8.29 Definition (Piecewise monotonic function.) A function f from an interval $[a, b]$ to $\mathbf{R}$ is *piecewise monotonic* if there are points $a_1, a_2, \cdots, a_n$ in $[a, b]$ with $a < a_1 < a_2 \cdots < a_n < b$ such that f is monotonic on each of the intervals $[a, a_1], [a_1, a_2], \cdots, [a_{n-1}, a_n], [a_n, b]$.

8.30 Example. The function whose graph is sketched below is piecewise monotonic.



piecewise monotonic function

8.31 Theorem. *Every piecewise monotonic function is integrable.*

Proof: This follows from corollary 8.23. |||

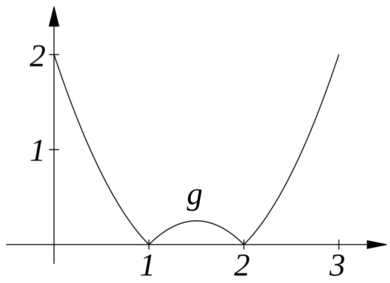
8.32 Exercise. Let

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ x - 1 & \text{if } 1 \le x \le 2. \end{cases}$$

Sketch the graph of f. Carefully explain why f is integrable, and find $\displaystyle\int_0^2 f$.

8.33 Example. Let $g(x) = |(x-1)(x-2)|$. Then

$$g(x) = \begin{cases} x^2 - 3x + 2 & \text{for } x \in [0, 1] \\ -x^2 + 3x - 2 & \text{for } x \in [1, 2] \\ x^2 - 3x + 2 & \text{for } x \in [2, 3]. \end{cases}$$



Hence g is integrable on $[0,3]$, and

$$\begin{aligned}
\int_0^3 g &= \int_0^1 (x^2-3x+2)dx - \int_1^2 (x^2-3x+2)dx + \int_2^3 (x^2-3x+2)dx \\
&= \left(\frac{1}{3} - 3\cdot\frac{1}{2} + 2\right) - \left(\frac{2^3-1^3}{3} - 3\cdot\frac{2^2-1^2}{2} + 2\right) \\
&\qquad + \left(\frac{3^3-2^3}{3} - 3\cdot\frac{3^2-2^2}{2} + 2\right) \\
&= \left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left(\frac{7}{3} - \frac{9}{2} + 2\right) + \left(\frac{19}{3} - \frac{15}{2} + 2\right) \\
&= \frac{13}{3} + \frac{-9}{2} + 2 = \frac{11}{6}
\end{aligned}$$

8.34 Exercise. Calculate the following integrals. Simplify your answers if you can.

a) $\displaystyle\int_0^2 |x^3-1|dx.$

b) $\displaystyle\int_a^b (x-a)(b-x)dx.$ Here $0 < a < b$.

c) $\displaystyle\int_a^b |(x-a)(b-x)|dx.$ Here $0 < a < b$.

d) $\displaystyle\int_0^1 (t^2-2)^3 dt.$

8.3 A Non-integrable Function

We will now give an example of a function that is not integrable. Let

$$S = \{\frac{m}{n} : m \in \mathbf{Z}, n \in \mathbf{Z}^+, \ \ m \text{ and } n \text{ are both odd}\}$$

$$T = \{\frac{m}{n}: m \in \mathbf{Z}, n \in \mathbf{Z}^+, \ \ m \text{ is even and } n \text{ is odd}\}.$$

Then $S \cap T = \emptyset$, since if $\dfrac{m}{n} = \dfrac{p}{q}$ where m, n, and q are odd and p is even, then $mq = np$ which is impossible since mq is odd and np is even.

8.35 Lemma. *Every interval (c, d) in $\mathbf{R}$ with $d - c > 0$ contains a point in S and a point in T.*

Proof: Since $d - c > 0$ we can choose an odd integer n such that $n > \dfrac{3}{d-c}$, i.e., $nd - nc > 3$. Since the interval (nc, nd) has length > 3, it contains at least two integers p, q, say $nc < p < q < nd$. If p and q are both odd, then there is an even integer between them, and if p and q are both even, there is an odd integer between them, so in all cases we can find a set of integers $\{r, s\}$ one of which is even and the other is odd such that $nc < r < s < nd$, i.e., $c < \dfrac{r}{n} < \dfrac{s}{n} < d$. Then $\dfrac{r}{n}$ and $\dfrac{s}{n}$ are two elements of (c, d) one of which is in S, and the other of which is in T. ‖

8.36 Example (A non-integrable function.) Let $D\colon [0,1] \to \mathbf{R}_{\geq 0}$ be defined by

$$D(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases} \tag{8.37}$$

I will find two partition-sample sequences $(\{P_n\}, \{S_n\})$ and $(\{P_n\}, \{T_n\})$ such that

$$\{\sum(D, P_n, T_n)\} \to 0$$

and

$$\{\sum(D, P_n, S_n)\} \to 1.$$

It then follows that D is not integrable. Let P_n be the regular partition of $[0, 1]$ into n equal subintervals.

$$P_n = \Big\{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\Big\}.$$

Let S_n be a sample for P_n such that each point in S_n is in S and let T_n be a sample for P_n such that each point in T_n is in T. (We can find such samples by lemma 8.35.) Then for all $n \in \mathbf{Z}^+$

$$\sum(D, P_n, S_n) = \sum_{i=1}^{n} D(s_n)(x_i - x_{i-1}) = 1$$

and

$$\sum(D, P_n, T_n) = \sum_{i=1}^{n} D(t_n)(x_i - x_{i-1}) = 0.$$

So $\lim\{\sum(D, P_n, S_n)\} = 1$ and $\lim\{\sum(D, P_n, T_n)\} = 0$. Both $(\{P_n\}, \{S_n\})$ and $(\{P_n\}, \{T_n\})$ are partition-sample sequences for $[0, 1]$, so it follows that D is not integrable.

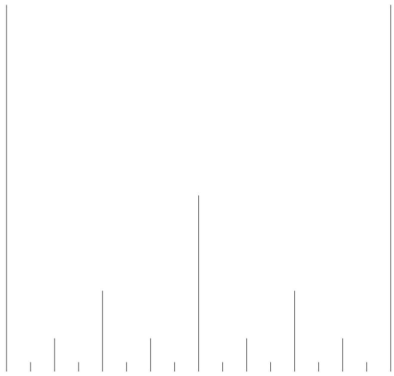
Our example of a non-integrable function is a slightly modified version of an example given by P. G. Lejeune Dirichlet (1805-1859) in 1837. Dirichlet's example was not presented as an example of a non-integrable function (since the definition of integrability in our sense had not yet been given), but rather as an example of how badly behaved a function can be. Before Dirichlet, functions that were this pathological had not been thought of as being functions. It was examples like this that motivated Riemann to define precisely what class of functions are well enough behaved so that we can prove things about them.

8.4 *The Ruler Function

8.38 Example (Ruler function.) We now present an example of an integrable function that is not monotonic on any interval of positive length. Define $R : [0, 1] \to \mathbf{R}$ by

$$R(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } x = 1 \\ \frac{1}{2^n} & \text{if } x = \frac{q}{2^n} \text{ where } q, n \in \mathbf{Z}^+ \text{ and } q \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

This formula defines $R(x)$ uniquely: If $\dfrac{q}{2^n} = \dfrac{p}{2^m}$ where p and q are odd, then $m = n$. (If $m > n$, we get $2^{m-n}q = p$, which says that an even number is odd.) The set $S_0^1 R$ under the graph of R is shown in the figure.



This set resembles the markings giving fractions of an inch on a ruler, which motivates the name *ruler function* for R. It is easy to see that R is not monotonic on any interval of length > 0. For each $p \in \mathbf{R}$ let $\delta_p : \mathbf{R} \to \mathbf{R}$ be defined by

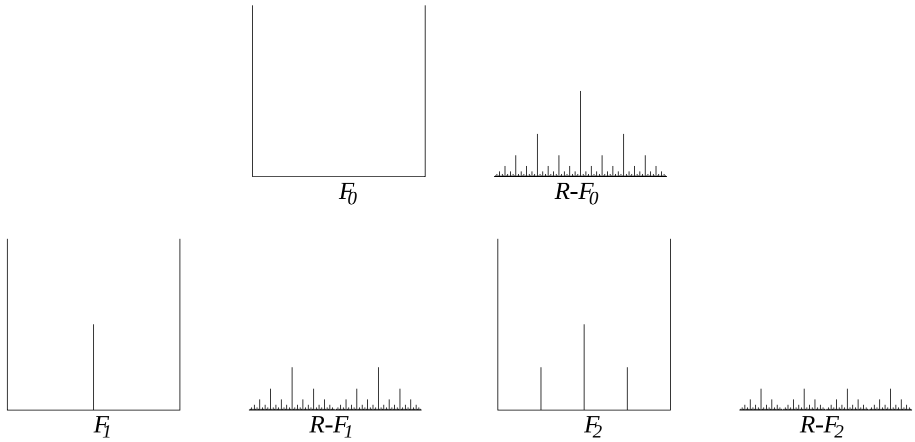
$$\delta_p(t) = \begin{cases} 1 & \text{if } p = t \\ 0 & \text{otherwise.} \end{cases}$$

We have seen that δ_p is integrable on any interval $[a, b]$ and $\int_a^b \delta_p = 0$. Now define a sequence of functions F_j by

$$\begin{aligned}
F_0 &= \delta_0 + \delta_1 \\
F_1 &= F_0 + \tfrac{1}{2}\delta_{\frac{1}{2}} \\
F_2 &= F_1 + \tfrac{1}{4}\delta_{\frac{1}{4}} + \tfrac{1}{4}\delta_{\frac{3}{4}} \\
&\vdots \\
F_n &= F_{n-1} + \tfrac{1}{2^n}\sum_{j=1}^{2^{n-1}} \delta_{\frac{2j-1}{2^n}}.
\end{aligned}$$

Each function F_j is integrable with integral 0 and

$$|R(x) - F_j(x)| \leq \frac{1}{2^{j+1}} \text{ for } 0 \leq x \leq 1.$$



I will now show that R is integrable.

Let $(\{P_n\},\{S_n\})$ be a partition-sample sequence for $[0,1]$. I'll show that $\{\sum(R,P_n,S_n)\}\to 0$.

Let ϵ be a generic element in $\mathbf{R}^+$. Observe that if $M\in\mathbf{Z}^+$ then

$$\left(\frac{1}{2^M}<\epsilon\right) \iff \left(M\ln(\frac{1}{2})<\ln(\epsilon)\right) \iff \left(M>\frac{\ln(\epsilon)}{\ln(\frac{1}{2})}\right).$$

Hence by the Archimedean property, we can choose $M\in\mathbf{Z}^+$ so that $\dfrac{1}{2^M}<\epsilon$. Then

$$\begin{aligned}\sum(R,P_n,S_n) &= \sum(R-F_M+F_M,P_n,S_n) && (8.39)\\ &= \sum(R-F_M,P_n,S_n)+\sum(F_M,P_n,S_n). && (8.40)\end{aligned}$$

Now since $0\le R(x)-F_M(x)\le \dfrac{1}{2^{M+1}}<\dfrac{1}{2}\epsilon$ for all $x\in[0,1]$, we have

$$\sum(R-F_M,P_n,S_n)\le\frac{1}{2^{M+1}}<\frac{1}{2}\epsilon \text{ for all } n\in\mathbf{Z}^+.$$

Since F_M is integrable and $\int F_M=0$, we have $\{\sum(F_M,P_n,S_n)\}\to 0$ so there is an $N\in\mathbf{Z}^+$ such that $|\sum(F_M,P_n,S_n)|<\dfrac{\epsilon}{2}$ for all $n\in\mathbf{Z}_{\ge N}$. By equation (8.40) we have

$$\begin{aligned}0 &\le \sum(R,P_n,S_n)=\sum(R-F_M,P_n,S_n)+\sum(F_M,P_n,S_n)\\ &< \frac{1}{2}\epsilon+\frac{1}{2}\epsilon=\epsilon \text{ for all } n\in\mathbf{Z}_{\ge N}.\end{aligned}$$

Hence $\{\sum(R, P_n, S_n)\} \to 0$, and hence R is integrable and $\int_0^1 R = 0$.

8.41 Exercise. Let R be the ruler function. We just gave a complicated proof that R is integrable and $\int_0^1 R = 0$. Explain why if you *assume* R is integrable, then it is easy to show that $\int_0^1 R = 0$.

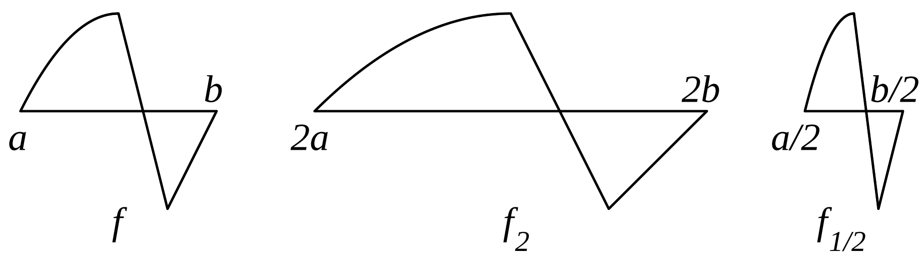
Also show that if you assume that the non-integrable function D in equation (8.37) is integrable then it is easy to show that $\int_0^1 D = 0$.

8.5 Change of Scale

8.42 Definition (Stretch of a function.) Let $[a, b]$ be an interval in $\mathbf{R}$, let $r \in \mathbf{R}^+$, and let $f\colon [a, b] \to \mathbf{R}$. We define a new function $f_r\colon [ra, rb] \to \mathbf{R}$ by

$$f_r(t) = f(\frac{t}{r}) \text{ for all } t \in [ra, rb].$$

If $t \in [ra, rb]$, then $\frac{t}{r} \in [a, b]$, so $f(\frac{t}{r})$ is defined.



The graph of f_r is obtained by stretching the graph of f by a factor of r in the horizontal direction, and leaving it unstretched in the vertical direction. (If $r < 1$ the stretch is actually a shrink.) I will call f_r the *stretch of f by r*.

8.43 Theorem (Change of scale for integrals.) *Let $[a,b]$ be an interval in $\mathbf{R}$ and let $r \in \mathbf{R}^+$. Let $f\colon [a,b] \to \mathbf{R}$ and let f_r be the stretch of f by r. If f is integrable on $[a,b]$ then f_r is integrable on $[ra,rb]$ and $\int_{ra}^{rb} f_r = r\int_a^b f$, i.e.,*

$$\int_{ra}^{rb} f(\frac{x}{r})dx = r\int_a^b f(x)dx. \tag{8.44}$$

Proof: Suppose f is integrable on $[a,b]$. Let $(\{P_n\},\{S_n\})$ be an arbitrary partition-sample sequence for $[ra,rb]$. If

$$P_n = \{x_0,\cdots,x_m\} \text{ and } S_n = \{s_1,\cdots,s_m\},$$

let

$$\frac{1}{r}P_n = \Big\{\frac{x_0}{r},\cdots,\frac{x_m}{r}\Big\} \text{ and } \frac{1}{r}S_n = \Big\{\frac{s_1}{r},\cdots,\frac{s_m}{r}\Big\}.$$

Then $\left(\left\{\frac{1}{r}P_n\right\},\left\{\frac{1}{r}S_n\right\}\right)$ is a partition-sample sequence for $[a,b]$, so $\{\sum(f,\frac{1}{r}P_n,\frac{1}{r}S_n)\} \to \int_a^b f$. Now

$$\begin{aligned}\sum(f_r,P_n,S_n) &= \sum_{i=1}^m f_r(s_i)(x_i - x_{i-1})\\ &= r\sum_{i=1}^m f(\frac{s_i}{r})\Big(\frac{x_i}{r} - \frac{x_{i-1}}{r}\Big) = r\sum\Big(f,\frac{1}{r}P_n,\frac{1}{r}S_n\Big)\end{aligned}$$

so

$$\lim\{\sum(f_r,P_n,S_n)\} = \lim\Big\{r\sum(f,\frac{1}{r}P_n,\frac{1}{r}S_n)\Big\} = r\int_a^b f.$$

This shows that f_r is integrable on $[ra,rb]$, and $\int_{ra}^{rb} f_r = r\int_a^b f$. |||

Remark: The notation f_r is not a standard notation for the stretch of a function, and I will not use this notation in the future. I will usually use the change of scale theorem in the form of equation (8.44), or in the equivalent form

$$\int_A^B g(rx)dx = \frac{1}{r}\int_{rA}^{rB} g(x)dx. \tag{8.45}$$

8.46 Exercise. Explain why formula (8.45) is equivalent to formula (8.44).

8.47 Example. We define π to be the area of the unit circle. Since the unit circle is carried to itself by reflections about the horizontal and vertical axes, we have

$$\pi = 4 \text{ (area (part of unit circle in the first quadrant))}.$$

Since points in the unit circle satisfy $x^2 + y^2 = 1$ or $y^2 = 1 - x^2$, we get

$$\pi = 4 \int_0^1 \sqrt{1 - x^2}\, dx.$$

We will use this result to calculate the area of a circle of radius a. The points on the circle with radius a and center $\mathbf{0}$ satisfy $x^2 + y^2 = a^2$, and by the same symmetry arguments we just gave

$$\begin{aligned}\text{area(circle of radius } a) &= 4\int_0^a \sqrt{a^2 - x^2}\, dx = 4\int_0^a a\sqrt{1 - \left(\frac{x}{a}\right)^2}\, dx \\ &= 4a \int_{a\cdot 0}^{a\cdot 1} \sqrt{1 - \left(\frac{x}{a}\right)^2}\, dx.\end{aligned}$$

By the change of scale theorem

$$\text{area(circle of radius } a) = 4aa \int_0^1 \sqrt{1 - x^2}\, dx = a^2\pi.$$

The formulas

$$\int_0^1 \sqrt{1 - x^2}\, dx = \frac{\pi}{4} \text{ and } \int_{-1}^1 \sqrt{1 - x^2}\, dx = \frac{\pi}{2}$$

or more generally

$$\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{\pi a^2}{4} \text{ and } \int_{-a}^a \sqrt{a^2 - x^2} = \frac{\pi a^2}{2},$$

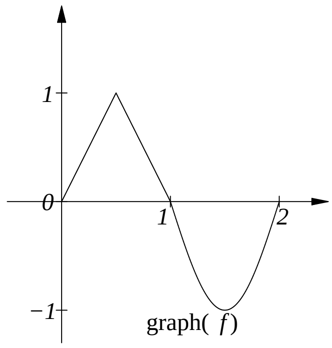
are worth remembering. Actually, these are cases of a formula you already know, since they say that the area of a circle of radius a is πa^2.

8.48 Exercise. Let a, b be positive numbers and let E_{ab} be the set of points inside the ellipse whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Calculate the area of E_{ab}.

8.49 Exercise. The figure shows the graph of a function f.



Let functions g, h, k, l, and m be defined by

a) $g(x) = f(\frac{x}{3})$.

b) $h(x) = f(3x)$.

c) $k(x) = f(\frac{x+3}{3})$.

d) $l(x) = f(3x+3)$.

e) $m(x) = 3f(\frac{x}{3})$.

Sketch the graphs of g,h,k, l, and m on different axes. Use the same scale for all of the graphs, and use the same scale on the x-axis and the y-axis,

8.50 Exercise. The value of $\int_0^1 \frac{1}{1+x^2}\,dx$ is .7854(approximately). Use this fact to calculate approximate values for

$$\int_0^a \frac{1}{a^2+x^2}dx \text{ and } \int_0^{\frac{1}{a}} \frac{1}{1+a^2x^2}dx$$

where $a \in \mathbf{R}^+$. Find numerical values for both of these integrals when $a = \frac{1}{4}$.

8.6 Integrals and Area

8.51 Theorem. *Let f be a piecewise monotonic function from an interval $[a,b]$ to $\mathbf{R}_{\geq 0}$. Then*
$$\int_a^b f = A_a^b f = \alpha(S_a^b f).$$

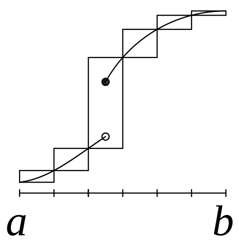
Proof: We already know this result for monotonic functions, and from this the result follows easily for piecewise monotonic functions. |||

Remark Theorem 8.51 is in fact true for all integrable functions from $[a,b]$ to $\mathbf{R}_{\geq 0}$, but the proof is rather technical. Since we will never need the result for functions that are not piecewise monotonic, I will not bother to make an assumption out of it.

8.52 Theorem. *Let $a, b \in \mathbf{R}$ and let $f\colon [a,b] \to \mathbf{R}$ be a piecewise monotonic function. Then the graph of f is a zero-area set.*

Proof: We will show that the theorem holds when f is monotonic on $[a,b]$. It then follows easily that the theorem holds when f is piecewise monotonic on $[a,b]$.

Suppose f is increasing on $[a,b]$. Let $n \in \mathbf{Z}^+$ and let $P = \{x_0, x_1, \cdots, x_n\}$ be the regular partition of $[a,b]$ into n equal subintervals.



Then
$$x_i - x_{i-1} = \frac{b-a}{n} \text{ for } 1 \leq i \leq n$$
and
$$\text{graph}(f) \subset \bigcup_{i=1}^{n} B\Big(x_{i-1}, x_i\colon f(x_{i-1}), f(x_i)\Big).$$

Hence

$$\begin{aligned}
0 \le \alpha\Big(\text{graph}(f)\Big) &\le \alpha\Big(\bigcup_{i=1}^{n} B(x_{i-1}, x_i \colon f(x_{i-1}), f(x_i))\Big)\\
&\le \sum_{i=1}^{n} \alpha\Big(B(x_{i-1}, x_i \colon f(x_{i-1}), f(x_i))\Big)\\
&= \sum_{i=1}^{n} (x_i - x_{i-1})\Big(f(x_i) - f(x_{i-1})\Big)\\
&= \sum_{i=1}^{n} \frac{b-a}{n}\Big(f(x_i) - f(x_{i-1})\Big)\\
&= \frac{b-a}{n}\sum_{i=1}^{n}\Big(f(x_i) - f(x_{i-1})\Big)\\
&= \frac{b-a}{n}\Big(f(b) - f(a)\Big).
\end{aligned}$$

Now $\Big\{\frac{b-a}{n}\Big(f(b)-f(a)\Big)\Big\} \to 0$, so it follows from the squeezing rule that the constant sequence $\Big\{\alpha\Big(\text{graph}(f)\Big)\Big\}$ converges to 0, and hence

$$\alpha\Big(\text{graph}(f)\Big) = 0. \ \|$$

Remark: Theorem 8.52 is actually valid for all integrable functions on $[a, b]$.

8.53 Theorem (Area between graphs.) *Let f, g be piecewise monotonic functions on an interval $[a, b]$ such that $g(x) \le f(x)$ for all $x \in [a, b]$. Let*

$$S = \{(x, y) \colon a \le x \le b \text{ and } g(x) \le y \le f(x)\}.$$

Then

$$\text{area}(S) = \int_a^b f(x) - g(x)\, dx.$$

Proof: Let M be a lower bound for g, so that

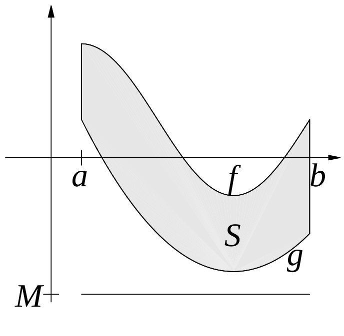
$$0 \le g(x) - M \le f(x) - M \text{ for all } x \in [a, b].$$

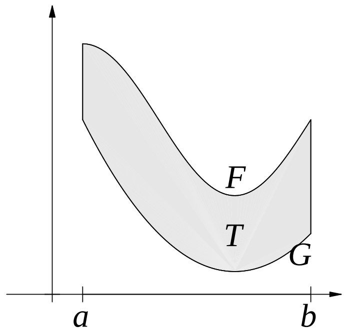
Let

$$F(x) = f(x) - M, \qquad G(x) = g(x) - M$$

for all $x \in [a, b]$, and let

$$T = \{(x, y) \colon a \le x \le b \text{ and } G(x) \le y \le F(x)\}.$$





Then

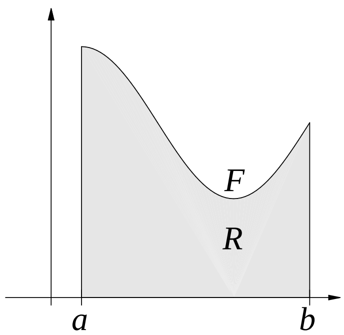
$$\begin{aligned}(x,y)\in T &\iff a\le x\le b \text{ and } G(x)\le y\le F(x)\\ &\iff a\le x\le b \text{ and } g(x)-M\le y\le f(x)-M\\ &\iff a\le x\le b \text{ and } g(x)\le y+M\le f(x)\\ &\iff (x,y+M)\in S\\ &\iff (x,y)+(0,M)\in S.\end{aligned}$$

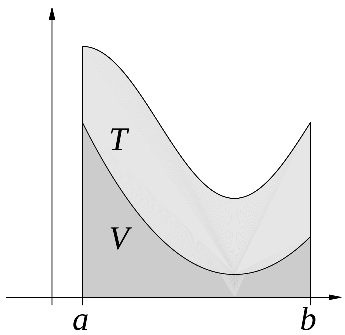
It follows from translation invariance of area that

$$\text{area}(S)=\text{area}(T).$$

Let

$$\begin{aligned}R &= \{(x,y)\colon a\le x\le b \text{ and } 0\le y\le F(x)\}=S_a^bF,\\ V &= \{(x,y)\colon a\le x\le b \text{ and } 0\le y\le G(x)\}=S_a^bG.\end{aligned}$$





Then $V\cup T=R$, and

$$V\cap T=\{(x,y)\colon a\le x\le b \text{ and } y=G(x)\}=\text{graph}(G).$$

It follows from theorem 8.52 that V and T are almost disjoint, so

$$\text{area}(R) = \text{area}(V \cup T) = \text{area}(V) + \text{area}(T),$$

and thus

$$\text{area}(T) = \text{area}(R) - \text{area}(V).$$

By theorem 8.51 we have

$$\text{area}(R) = \text{area}(S_a^b F) = \int_a^b F(x)\,dx$$

and

$$\text{area}(V) = \text{area}(S_a^b G) = \int_a^b G(x)\,dx.$$

Thus

$$\begin{aligned}
\text{area}(S) &= \text{area}(T) = \text{area}(R) - \text{area}(V)\\
&= \int_a^b F(x)\,dx - \int_a^b G(x)\,dx\\
&= \int_a^b \Big(F(x) - G(x)\Big)\,dx\\
&= \int_a^b \bigg(f(x) - M - \Big(g(x) - M\Big)\bigg)\,dx\\
&= \int_a^b f(x) - g(x)\,dx. \ \|
\end{aligned}$$

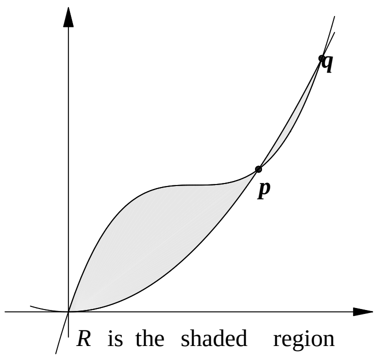
Remark: Theorem 8.53 is valid for all integrable functions f and g. This follows from our proof and the fact that theorems 8.51 and 8.52 are both valid for all integrable functions.

8.54 Example. We will find the area of the set R in the figure, which is bounded by the graphs of f and g where

$$f(x) = \frac{1}{2}x^2$$

and

$$g(x) = x^3 - 3x^2 + 3x.$$



Now

$$\begin{aligned} g(x) - f(x) &= x^3 - 3x^2 + 3x - \frac{1}{2}x^2 = x^3 - \frac{7}{2}x^2 + 3x \\ &= x(x^2 - \frac{7}{2}x + 3) = x(x-2)(x - \frac{3}{2}). \end{aligned}$$

Hence

$$(g(x) = f(x)) \iff (g(x) - f(x) = 0) \iff \left(x \in \left\{0, \frac{3}{2}, 2\right\}\right).$$

It follows that the points $\mathbf{p}$ and $\mathbf{q}$ in the figure are

$$\mathbf{p} = (\frac{3}{2}, f(\frac{3}{2})) = (\frac{3}{2}, \frac{9}{8}) \text{ and } \mathbf{q} = (2, f(2)) = (2, 2).$$

Also, since $x(x-2) \le 0$ for all $x \in [0, 2]$,

$$g(x) - f(x) \ge 0 \iff x - \frac{3}{2} \le 0 \iff x \le \frac{3}{2}.$$

(This is clear from the picture, assuming that the picture is accurate.) Thus

$$\begin{aligned} \text{area}(R) &= \int_0^{\frac{3}{2}} (g - f) + \int_{\frac{3}{2}}^2 (f - g) \\ &= \int_0^{\frac{3}{2}} (x^3 - \frac{7}{2}x^2 + 3x)dx - \int_{\frac{3}{2}}^2 (x^3 - \frac{7}{2}x^2 + 3x)dx \\ &= \left(\frac{(\frac{3}{2})^4 - 0^4}{4}\right) - \frac{7}{2}\left(\frac{(\frac{3}{2})^3 - 0^3}{3}\right) + 3\left(\frac{(\frac{3}{2})^2 - 0^2}{2}\right) \\ &\qquad - \left(\frac{2^4 - (\frac{3}{2})^4}{4}\right) + \frac{7}{2}\left(\frac{2^3 - (\frac{3}{2})^3}{3}\right) - 3\left(\frac{2^2 - (\frac{3}{2})^2}{2}\right). \end{aligned}$$

We have now found the area, but the answer is not in a very informative form. It is not clear whether the number we have found is positive. It would be reasonable to use a calculator to simplify the result, but my experience with calculators is that I am more likely to make an error entering this into my calculator than I am to make an error by doing the calculation myself, so I will continue. I notice that three terms in the answer are repeated twice, so I have

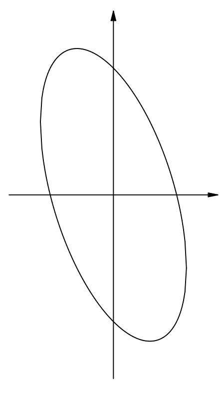
$$\begin{aligned}
\text{area}(R) &= 2\left(\frac{81}{64} - \frac{63}{16} + \frac{27}{8}\right) - 4 + \frac{28}{3} - 6 \\
&= \frac{81}{32} - \frac{63}{8} + \frac{27}{4} - \frac{2}{3} \\
&= (2 + \frac{17}{32}) - (8 - \frac{1}{8}) + (6 + \frac{3}{4}) - \frac{2}{3} \\
&= \frac{17}{32} + \frac{1}{8} + \frac{3}{4} - \frac{2}{3} = \frac{21}{32} + \frac{1}{12} = \frac{63+8}{96} = \frac{71}{96}.
\end{aligned}$$

Thus the area is about 0.7 From the sketch I expect the area to be a little bit smaller than 1, so the answer is plausible.

8.55 Exercise. The curve whose equation is

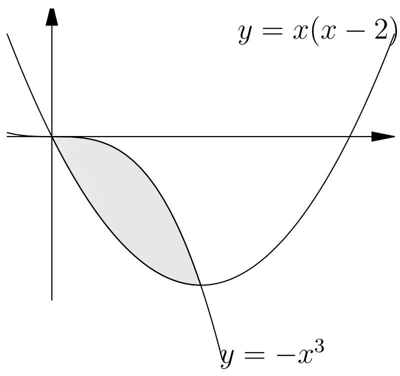
$$y^2 + 2xy + 2x^2 = 4 \tag{8.56}$$

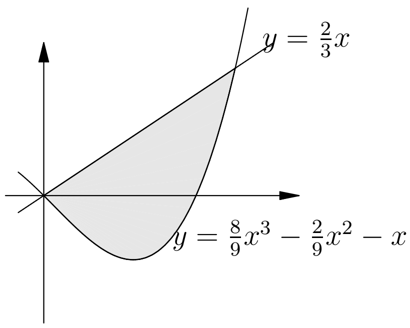
is shown in the figure. Find the area enclosed by the curve.



(The set whose area we want to find is bounded by the graphs of the two functions. You can find the functions by considering equation (8.56) as a quadratic equation in y and solving for y as a function of x.)

8.57 Exercise. Find the areas of the two sets shaded in the figures below:





8.58 Exercise. Find the area of the shaded region.