Chapter 7

Still More Area Calculations

7.1 Area Under a Monotonic Function

7.1 Theorem. *Let f be a monotonic function from the interval $[a, b]$ to $\mathbf{R}_{\geq 0}$. Let $\{P_n\}$ be a sequence of partitions of $[a, b]$ such that $\{\mu(P_n)\} \to 0$, and let*

$$A_a^b f = \alpha\{(x, y) \in \mathbf{R}^2 : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}$$

Then

$$\{\alpha\Big(I_a^b(f, P_n)\Big)\} \to A_a^b f$$

and

$$\{\alpha\Big(O_a^b(f, P_n)\Big)\} \to A_a^b f.$$

(The notation here is the same as in theorem 5.40 and exercise 5.47.)

Proof: We noted in theorem 5.40 and exercise 5.47 that

$$0 \leq \alpha\Big(O_a^b(f, P_n)\Big) - \alpha\Big(I_a^b(f, P_n)\Big) \leq \mu(P_n) \cdot |f(b) - f(a)|. \tag{7.2}$$

Since

$$\begin{aligned} \lim \{\mu(P_n) \cdot |f(b) - f(a)|\} &= |f(b) - f(a)| \lim\{\mu(P_n)\} \\ &= |f(b) - f(a)| \cdot 0 = 0, \end{aligned}$$

we conclude from the squeezing rule that

$$\lim \Big\{\alpha\Big(O_a^b(f, P_n)\Big) - \alpha\Big(I_a^b(f, P_n)\Big)\Big\} = 0. \tag{7.3}$$

We also have by (5.43) that

$$\alpha\Big(I_a^b(f,P_n)\Big) \leq A_a^b f \leq \alpha\Big(O_a^b(f,P_n)\Big),$$

so that

$$0 \leq A_a^b f - \alpha\Big(I_a^b(f,P_n)\Big) \leq \alpha\Big(O_a^b(f,P_n)\Big) - \alpha\Big(I_a^b(f,P_n)\Big).$$

By (7.3) and the squeezing rule

$$\lim\Big\{A_a^b f - \alpha\Big(I_a^b(f,P_n)\Big)\Big\} = 0,$$

and hence

$$\lim\Big\{\alpha\Big(I_a^b(f,P_n)\Big)\Big\} = A_a^b f.$$

Also,

$$\begin{aligned}
\lim\Big\{\alpha\Big(O_a^b(f,P_n)\Big)\Big\} &= \lim\Big\{\alpha\Big(I_a^b(f,P_n)\Big) + \Big(\alpha(O_a^b(f,P_n)) - \alpha(I_a^b(f,P_n))\Big)\Big\} \\
&= \lim\Big\{\alpha\Big(I_a^b(f,P_n)\Big)\Big\} \\
&\qquad + \lim\Big\{\Big(\alpha(O_a^b(f,P_n)) - \alpha(I_a^b(f,P_n))\Big)\Big\} \\
&= A_a^b f + 0 = A_a^b f. \ \|\!|
\end{aligned}$$

7.4 Definition (Riemann sum). Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition for an interval $[a,b]$. A *sample* for P is a finite sequence $S = \{s_1, s_2, \cdots, s_n\}$ of numbers such that $s_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$. If f is a function from $[a,b]$ to $\mathbf{R}$, and P is a partition for $[a,b]$ and S is a sample for P, we define

$$\sum(f,P,S) = \sum_{i=1}^{n} f(s_i)(x_i - x_{i-1})$$

and we call $\sum(f,P,S)$ a *Riemann sum* for f, P and S. We will sometimes write $\sum([f(t)],P,S)$ instead of $\sum(f,P,S)$.

7.5 Example. If f is an increasing function from $[a,b]$ to $\mathbf{R}_{\geq 0}$, and $P = \{x_0, \cdots, x_n\}$ is a partition of $[a,b]$, and $S_l = \{x_0, \cdots, x_{n-1}\}$, then

$$\sum(f,P,S_l) = \alpha(I_a^b(f,P)).$$

If $S_r = \{x_1, x_2, \cdots, x_n\}$, then

$$\sum(f,P,S_r) = \alpha(O_a^b(f,P)).$$

If $S_m = \{\frac{x_0 + x_1}{2}, \cdots, \frac{x_{n-1} + x_n}{2}\}$ then

$$\sum(f, P, S_m) = \sum_{i=1}^{n} f\Big(\frac{x_{i-1} + x_i}{2}\Big)(x_i - x_{i-1})$$

is some number between $\alpha(I_a^b(f, P))$ and $\alpha(O_a^b(f, P))$.

7.6 Theorem (Area theorem for monotonic functions.) *Let f be a monotonic function from the interval $[a, b]$ to $\mathbf{R}_{\geq 0}$. Then for every sequence $\{P_n\}$ of partitions of $[a, b]$ such that $\{\mu(P_n)\} \to 0$, and for every sequence $\{S_n\}$ where S_n is a sample for P_n, we have*

$$\{\sum(f, P_n, S_n)\} \to A_a^b f.$$

Proof: We will consider the case where f is increasing. The case where f is decreasing is similar.

For each partition $P_n = \{x_0, \cdots, x_m\}$ and sample $S_n = \{s_1, \cdots, s_m\}$, we have for $1 \leq i \leq m$

$$\begin{aligned} x_{i-1} \leq s_i \leq x_i &\implies f(x_{i-1}) \leq f(s_i) \leq f(x_i) \\ &\implies f(x_{i-1})(x_i - x_{i-1}) \leq f(s_i)(x_i - x_{i-1}) \leq f(x_i)(x_i - x_{i-1}). \end{aligned}$$

Hence

$$\sum_{i=1}^{m} f(x_{i-1})(x_i - x_{i-1}) \leq \sum_{i=1}^{m} f(s_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{m} f(x_i)(x_i - x_{i-1}),$$

i.e.,

$$\alpha\Big(I_a^b(f, P_n)\Big) \leq \sum(f, P_n, S_n) \leq \alpha\Big(O_a^b(f, P_n)\Big).$$

By theorem 7.1 we have

$$\{\alpha\Big(I_a^b(f, P_n)\Big)\} \to A_a^b f,$$

and

$$\{\alpha\Big(O_a^b(f, P_n)\Big)\} \to A_a^b f,$$

so by the squeezing rule,

$$\{\sum(f, P_n, S_n)\} \to A_a^b f.$$

7.2 Calculation of Area under Power Functions

7.7 Lemma. *Let r be a rational number such that $r \neq -1$. Let a be a real number with $a > 1$. Then*

$$A_1^a[t^r] = (a^{r+1} - 1) \lim \left\{ \frac{a^{\frac{1}{n}} - 1}{a^{\frac{r+1}{n}} - 1} \right\}.$$

(For the purposes of this lemma, we will assume that the limit exists. In theorem 7.10 we will prove that the limit exists.)

Proof: Let n be a generic element of $\mathbf{Z}^+$. To simplify the notation, I will write

$$p = a^{\frac{1}{n}}, \text{ (so } p > 1).$$

Let

$$P_n = \{1, a^{\frac{1}{n}}, a^{\frac{2}{n}}, \cdots, a^{\frac{n}{n}}\} = \{1, p, p^2, \cdots, p^n\} = \{x_0, x_1, x_2, \cdots, x_n\}$$

and let

$$S_n = \{1, p, p^2, \cdots, p^{n-1}\} = \{s_1, s_2, s_3 \cdots, s_n\}.$$

Then for $1 \leq i \leq n$

$$x_i - x_{i-1} = p^i - p^{i-1} = p^{i-1}(p-1),$$

so

$$\mu(P_n) = p^{n-1}(p-1) \leq p^n(p-1) = a\left(a^{\frac{1}{n}} - 1\right).$$

It follows by the nth root rule (theorem 6.48) that $\{\mu(P_n)\} \to 0$. Hence it follows from theorem 7.6 that

$$A_1^a[t^r] = \lim \left(\sum([t^r], P_n, S_n) \right). \tag{7.8}$$

Now

$$\begin{aligned} \sum([t^r], P_n, S_n) &= \sum_{i=1}^{n} s_i^r (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (p^{(i-1)})^r p^{i-1}(p-1) \end{aligned}$$

$$\begin{aligned}
&= (p-1)\sum_{i=1}^{n}\left(p^{r+1}\right)^{(i-1)} && (7.9)\\
&= (p-1)\left(\frac{p^{(r+1)n}-1}{p^{r+1}-1}\right) = \left((p^n)^{r+1}-1\right)\left(\frac{p-1}{p^{r+1}-1}\right)\\
&= \left(a^{r+1}-1\right)\frac{(a^{\frac{1}{n}}-1)}{(a^{\frac{r+1}{n}}-1)}.
\end{aligned}$$

Here we have used the formula for a finite geometric series. Thus, from (7.8)

$$\begin{aligned}
A_1^a[t^r] &= \lim\left\{(a^{r+1}-1)\frac{(a^{\frac{1}{n}}-1)}{(a^{\frac{r+1}{n}}-1)}\right\}\\
&= (a^{r+1}-1)\lim\left\{\frac{a^{\frac{1}{n}}-1}{a^{\frac{r+1}{n}}-1}\right\}. \parallel
\end{aligned}$$

Now we want to calculate the limit appearing in the previous lemma. In order to do this it will be convenient to prove a few general limit theorems.

7.10 Theorem. *Let $\{x_n\}$ be a sequence of positive numbers such that $\{x_n\}\to 1$ and $x_n\neq 1$ for all $n\in\mathbf{Z}^+$. Let β be any rational number. Then*

$$\left\{\frac{x_n^\beta-1}{x_n-1}\right\}\to\beta.$$

Proof: Suppose $x_n\neq 1$ for all n, and $\{x_n\}\to 1$.
Case 1: Suppose $\beta=0$. Then the conclusion clearly follows.
Case 2: Suppose $\beta\in\mathbf{Z}^+$. Then by the formula for a geometric series

$$\frac{x_n^\beta-1}{x_n-1}=1+x_n+\cdots+x_n^{\beta-1}.$$

By the sum theorem and many applications of the product theorem we conclude that

$$\begin{aligned}
\lim\left\{\frac{x_n^\beta-1}{x_n-1}\right\} &= \lim\{1\}+\lim\{x_n\}+\cdots+\lim\{x_n^{\beta-1}\}\\
&= 1+1+1+\cdots+1\\
&= \beta.
\end{aligned}$$

Case 3: Suppose $\beta \in \mathbf{Z}^-$. Let $\gamma = -\beta$. Then $\gamma \in \mathbf{Z}^+$, so by Case 2 we get

$$\begin{aligned}
\lim\Big\{\frac{x_n^\beta - 1}{x_n - 1}\Big\} &= \lim\Big\{\frac{x_n^\gamma(x_n^\beta - 1)}{x_n^\gamma(x_n - 1)}\Big\} = \lim\Big\{\frac{1 - x_n^\gamma}{x_n^\gamma(x_n - 1)}\Big\} \\
&= \lim\Big\{\frac{1}{-x_n^\gamma}\Big(\frac{x_n^\gamma - 1}{x_n - 1}\Big)\Big\} \\
&= \lim\Big\{\frac{1}{-x_n^\gamma}\Big\}\lim\Big\{\frac{x_n^\gamma - 1}{x_n - 1}\Big\} \\
&= \frac{1}{-1}\cdot\gamma = -\gamma = \beta.
\end{aligned}$$

Case 4: Suppose $\beta = \dfrac{p}{q}$ where $q \in \mathbf{Z}^+$ and $p \in \mathbf{Z}$. Let $y_n = x_n^{\frac{1}{q}}$. Then

$$\frac{x_n^\beta - 1}{x_n - 1} = \frac{x_n^{\frac{p}{q}} - 1}{x_n - 1} = \frac{y_n^p - 1}{y_n^q - 1} = \frac{\Big(\dfrac{y_n^p - 1}{y_n - 1}\Big)}{\Big(\dfrac{y_n^q - 1}{y_n - 1}\Big)}.$$

Now if we could show that $\{y_n\} \to 1$, it would follow from this formula that

$$\lim\Big\{\frac{x_n^\beta - 1}{x_n - 1}\Big\} = \frac{\lim\Big\{\frac{y_n^p - 1}{y_n - 1}\Big\}}{\lim\Big\{\frac{y_n^q - 1}{y_n - 1}\Big\}} = \frac{p}{q} = \beta.$$

The next lemma shows that $\{y_n\} \to 1$ and completes the proof of theorem 7.10.

7.11 Lemma. *Let $\{x_n\}$ be a sequence of positive numbers such that $\{x_n\} \to 1$, and $\{x_n\} \neq 1$ for all $n \in \mathbf{Z}^+$. Then for each q in $\mathbf{Z}^+$, $\{x_n^{\frac{1}{q}}\} \to 1$.*

Proof: Let $\{x_n\}$ be a sequence of positive numbers such that $\{x_n\} \to 1$. Let $y_n = x_n^{\frac{1}{q}}$ for each n in $\mathbf{Z}^+$. We want to show that $\{y_n\} \to 1$. By the formula for a finite geometric series

$$1 + y_n + \cdots + y_n^{q-1} = \frac{(1 - y_n^q)}{1 - y_n} = \frac{(1 - x_n)}{1 - y_n}$$

so

$$(1 - y_n) = \frac{(1 - x_n)}{1 + y_n + \cdots + y_n^{q-1}}.$$

Now

$$0 \leq |1 - y_n| = \frac{|1 - x_n|}{|1 + y_n + \cdots + y_n^{q-1}|} = \frac{|1 - x_n|}{1 + y_n + \cdots + y_n^{q-1}} \leq |1 - x_n|.$$

Since $\{x_n\} \to 1$, we have $\lim\{|1-x_n|\} = 0$, so by the squeezing rule $\lim\{|1-y_n|\} = 0$, and hence

$$\lim\{y_n\} = 1. \ \|$$

7.12 Lemma (Calculation of $A_1^b[t^r]$.) *Let b be a real number with $b > 1$, and let $r \in \mathbf{Q} \setminus \{-1\}$. Then*

$$A_1^b[t^r] = \frac{b^{r+1} - 1}{r + 1}.$$

Proof: By lemma 7.7,

$$A_1^b[t^r] = (b^{r+1} - 1) \lim \left\{ \frac{b^{\frac{1}{n}} - 1}{b^{\frac{r+1}{n}} - 1} \right\}.$$

By theorem 7.10,

$$\lim \left\{ \frac{b^{\frac{1}{n}} - 1}{b^{\frac{r+1}{n}} - 1} \right\} = \lim \left\{ \frac{1}{\frac{b^{\frac{r+1}{n}} - 1}{b^{\frac{1}{n}} - 1}} \right\} = \frac{\lim\{1\}}{\lim \left\{ \frac{b^{\frac{r+1}{n}} - 1}{b^{\frac{1}{n}} - 1} \right\}} = \frac{1}{r + 1},$$

and putting these results together, we get

$$A_1^b[t^r] = \frac{b^{r+1} - 1}{r + 1}. \ \|$$

7.13 Lemma. *Let $r \in \mathbf{Q}$, and let $a, c \in \mathbf{R}^+$, with $1 < c$. Then*

$$A_a^{ca}[t^r] = a^{r+1} A_1^c[t^r].$$

Proof: If

$$P = \{x_0, x_1, \ldots, x_n\}$$

is a partition of $[1, c]$, let

$$aP = \{ax_0, ax_1, \ldots, ax_n\}$$

be the partition of $[a, ca]$ obtained by multiplying the points of P by a. Then

$$\mu(aP) = a\mu(P). \tag{7.14}$$

If

$$S = \{s_1, s_2, \ldots, s_n\}$$

is a sample for P, let

$$aS = \{as_1, as_2, \ldots, as_n\}$$

be the corresponding sample for aP. Then

$$\begin{aligned}
\sum([t^r], aP, aS) &= \sum_{i=1}^{n} (as_i)^r (ax_i - ax_{i-1}) \\
&= \sum_{i=1}^{n} a^r s_i^r a(x_i - x_{i-1}) \\
&= a^{r+1} \sum_{i=1}^{n} s_i^r (x_i - x_{i-1}) \\
&= a^{r+1} \sum([t^r], P, S).
\end{aligned}$$

Let $\{P_n\}$ be a sequence of partitions of $[1, c]$ such that $\{\mu(P_n)\} \to 0$, and for each $n \in \mathbf{Z}^+$ let S_n be a sample for P_n. It follows from (7.14) that $\{\mu(aP_n)\} \to 0$. By the area theorem for monotonic functions (theorem 7.6), we have

$$\left\{\sum([t^r], P_n, S_n)\right\} \to A_1^c[t^r] \text{ and } \left\{\sum([t^r], aP_n, aS_n)\right\} \to A_a^{ca}[t^r].$$

Thus

$$\begin{aligned}
A_a^{ca}[t^r] &= \lim\{\sum([t^r], aP_n, aS_n)\} \\
&= \lim\{a^{r+1} \sum([t^r], P_n, S_n)\} = a^{r+1} \lim\{\sum([t^r], P_n, S_n)\} \\
&= a^{r+1} A_1^c[t^r]. \ \|
\end{aligned}$$

7.15 Theorem (Calculation of $A_a^b[t^r]$.) *Let $a, b \in \mathbf{R}^+$ with $a < b$, and let $r \in \mathbf{Q}$. Then*

$$A_a^b[t^r] = \begin{cases} \frac{b^{r+1} - a^{r+1}}{r+1} & \text{if } r \neq -1 \\ \ln(b) - \ln(a) & \text{if } r = -1. \end{cases}$$

Proof: The result for the case $r = -1$ was proved in theorem 5.76. The case $r \neq -1$ is done in the following exercise.

7.16 Exercise. Use the two previous lemmas to prove theorem 7.15 for the case $r \neq -1$.

Remark: In the proof of lemma 7.7, we did not use the assumption $r \neq -1$ until line (7.9). For $r = -1$ equation (7.9) becomes

$$\sum([t^{-1}], P_n, S_n) = n(a^{\frac{1}{n}} - 1).$$

Since in this case $\{\sum([t^{-1}], P_n, S_n)\} \to A_1^a[\frac{1}{t}] = \ln(a)$,we conclude that

$$\lim\{n(a^{\frac{1}{n}} - 1)\} = \ln(a) \text{ for all } a > 1. \tag{7.17}$$

This formula give us method of calculating logarithms by taking square roots. We know $2^n(a^{\frac{1}{2^n}} - 1)$ will be near to $\ln(a)$ when n is large, and $a^{\frac{1}{2^n}}$ can be calculated by taking n successive square roots. On my calculator, I pressed the following sequence of keys

$$2\underbrace{\surd\surd\cdots\surd}_{15 \text{ times}} -1 \;=\; \underbrace{\times 2 \quad \times 2 \quad \cdots \quad \times 2}_{15 \text{ times}} \;=$$

and got the result 0.693154611. My calculator also says that
$\ln(2) = 0.69314718$. It appears that if I know how to calculate square roots, then I can calculate logarithms fairly easily.

7.18 Exercise. Let r be a non-negative rational number, and let $b \in \mathbf{R}^+$. Show that

$$A_0^b[t^r] = \frac{b^{r+1}}{r+1}.$$

Where in your proof do you use the fact that $r \geq 0$?