Chapter 3

Propositions and Functions

In this chapter we will introduce some general mathematical ideas and notation that will be useful in the following chapters.

3.1 Propositions

3.1 Definition (Proposition.) A *proposition* is a statement that is either true or false. I will sometimes write a proposition inside of quotes (“ ”), when I want to emphasize where the proposition begins and ends.

3.2 Examples.
If $P_1 =$ “$1 + 1 = 2$”, then P_1 is a true proposition.
If $P_2 =$ “$1 + 1 = 3$”, then P_2 is a false proposition.
If $P_3 =$ “2 is an even number”, then P_3 is a true proposition.
If $P_4 =$ “7 is a lucky number”, then I will not consider P_4 to be a proposition (unless *lucky number* has been defined.)

3.3 Definition (And, or, not.) Suppose that P and Q are propositions. Then we can form new propositions denoted by “P and Q”, “P or Q”, and “not P”.
“P and Q” is true if and only if both of P, Q are true.
“P or Q” is true if and only if at least one of P, Q is true.
“not P” is true if and only if P is false.
Observe that in mathematics, “or” is always assumed to be inclusive or: If “P” and “Q” are both true, then “P or Q” is true.

3.4 Examples.
"$1 + 1 = 2$ and $1 + 1 = 3$" is false.
"$1 + 1 = 2$ or $1 + 1 = 3$" is true.
"$1 + 1 = 2$ or $2 + 2 = 4$" is true.
"not(not P)" is true if and only if P is true.

For each element x of $\mathbf{Q}$ let $R(x)$ be the proposition "$x^2 + 5x + 6 = 0$". Thus $R(-3) =$ "$(-3)^2 + 5 \cdot (-3) + 6 = 0$", so $R(-3)$ is true, while $R(0) =$ "$0^2 + 5 \cdot 0 + 6 = 0$", so $R(0)$ is false. Here I consider R to be a rule which assigns to each element x of $\mathbf{Q}$ a proposition $R(x)$.

3.5 Definition (Proposition form.) Let S be a set. A rule P that assigns to each element x of S a unique proposition $P(x)$ is called a *proposition form over* S.

Thus the rule R defined in the previous paragraph is a proposition form over $\mathbf{Q}$. Note that a proposition form is neither true nor false, i.e. a proposition form is not a proposition.

3.6 Definition ($\Longleftrightarrow$, Equivalent propositions.) Let P, Q be two propositions. We say that "P is *equivalent to* Q" if either (P, Q are both true) or (P, Q are both false). Thus every proposition is equivalent either to "$1+1 = 2$" or to "$1+1 = 3$." We write "$P \Longleftrightarrow Q$" as an abbreviation for "P is equivalent to Q." If P, Q are propositions, then "$P \Longleftrightarrow Q$" is a proposition, and
"$P \Longleftrightarrow Q$" is true if and only if ((P, Q are both true) or (P, Q are both false)).

Ordinarily one would not make a statement like
"$(1 + 1 = 2) \Longleftrightarrow$ (4421 is a prime number)"
even though this is a true proposition. One writes "$P \Longleftrightarrow Q$" in an argument, only when the person reading the argument can be expected to see the equivalence of the two statements P and Q.

If P, Q, R and S are propositions,then

$$P \Longleftrightarrow Q \Longleftrightarrow R \Longleftrightarrow S \tag{3.7}$$

is an abbreviation for

$$((P \Longleftrightarrow Q) \text{ and } (Q \Longleftrightarrow R)) \text{ and } (R \Longleftrightarrow S).$$

Thus if we know that (3.7) is true, then we can conclude that $P \Longleftrightarrow S$ is true. The statement "$P \Longleftrightarrow Q$" is sometimes read as " P if and only if Q".

3.8 Example. Find all real numbers x such that

$$x^2 - 5x + 6 = 0. \tag{3.9}$$

Let x be an arbitrary real number. Then

$$\begin{aligned} x^2 - 5x + 6 = 0 &\iff (x-2)(x-3) = 0 \\ &\iff ((x-2) = 0) \text{ or } ((x-3) = 0) \\ &\iff (x = 2) \text{ or } (x = 3). \end{aligned}$$

Thus the set of all numbers that satisfy equation (3.9) is $\{2,3\}$. $\|$

3.10 Definition ($\Longrightarrow$, Implication.) If P and Q are propositions then we say "P implies Q" and write "$P \Longrightarrow Q$", if the truth of Q follows from the truth of P. We make the convention that if P is false then $(P \Longrightarrow Q)$ is true for all propositions Q, and in fact that

$$(P \Longrightarrow Q) \text{ is true } \textit{unless } (P \text{ is true and } Q \text{ is false}). \tag{3.11}$$

Hence for all propositions P and Q

$$(P \Longrightarrow Q) \iff (Q \text{ or } \quad \text{not}(P)). \tag{3.12}$$

3.13 Example. For every element x in $\mathbf{Q}$

$$x = 2 \Longrightarrow x^2 = 4. \tag{3.14}$$

In particular, the following statements are all true.

$$2 = 2 \Longrightarrow 2^2 = 4. \tag{3.15}$$

$$-2 = 2 \Longrightarrow (-2)^2 = 4. \tag{3.16}$$

$$3 = 2 \Longrightarrow 3^2 = 4. \tag{3.17}$$

In proposition 3.16, P is false, Q is true, and $P \Longrightarrow Q$ is true.
In proposition 3.17, P is false, Q is false, and $P \Longrightarrow Q$ is true.

The usual way to prove $P \Longrightarrow Q$ is to assume that P is true and show that then Q must be true. This is sufficient by our convention in (3.11).

If P and Q are propositions, then "$P \Longrightarrow Q$" is also a proposition, and

$$(P \iff Q) \text{ is equivalent to } (P \Longrightarrow Q \text{ and } \; Q \Longrightarrow P) \tag{3.18}$$

(the right side of (3.18) is true if and only if P, Q are both true or both false.) An alternate way of writing "$P \Longrightarrow Q$" is "if P then Q".

We will not make much use of the idea of two propositions being equal. Roughly, two propositions are equal if and only if they are word for word the same. Thus "$1+1=2$" and "$2=1+1$" are not equal propositions, although they are equivalent. The only time I will use an "=" sign between propositions is in definitions. For example, I might define a proposition form P over $\mathbf{N}$ by saying

for all $n \in \mathbf{N}$, $P(n) =$"$n+1=2$",

or

for all $n \in \mathbf{N}$, $P(n) = [n+1=2]$.

The definition we have given for "implies" is a matter of convention, and there is a school of contemporary mathematicians (called constructivists) who define $P \Longrightarrow Q$ to be true only if a "constructive" argument can be given that the truth of Q follows from the truth of P. For the constructivists, some of the propositions of the sort we use are neither true nor false, and some of the theorems we prove are not provable (or disprovable). A very readable description of the constructivist point of view can be found in the article *Schizophrenia in Contemporary Mathematics*[10, pages 1–10].

3.19 Exercise.

a) Give examples of propositions P, Q such that "$P \Longrightarrow Q$" and "$Q \Longrightarrow P$" are both true, or else explain why no such examples exist.

b) Give examples of propositions R, S such that "$R \Longrightarrow S$" and "$S \Longrightarrow R$" are both false, or explain why no such examples exist.

c) Give examples of propositions T, V such that "$T \Longrightarrow V$" is true but "$V \Longrightarrow T$" is false, or explain why no such examples exist.

3.20 Exercise. Let P, Q be two propositions. Show that the propositions "$P \Longrightarrow Q$" and " not$Q \Longrightarrow$ not$P$" are equivalent. ("not$Q \Longrightarrow$ not$P$" is called the *contrapositive* of the statement "$P \Longrightarrow Q$".)

3.21 Exercise. Which of the proposition forms below are true for all real numbers x? If a proposition form is not true for all real numbers x, give a number for which it is false.

a) $x = 1 \Longrightarrow x^2 = 1$.

b) $x^2 = 1 \implies x = 1$.

c) $x < \frac{1}{2} \implies 2x < 1$.

d) $2 < \frac{1}{x} \iff 2x < 1$. (Here assume $x \neq 0$.)

e) $x < 1 \implies x + 1 < 3$.

f) $x < 1 \iff x + 1 < 3$.

g) $x \leq 1 \implies x < 1$.

h) $x < 1 \implies x \leq 1$.

3.22 Exercise. Both of the arguments A and B given below are faulty, although one of them leads to a correct conclusion. Criticize both arguments, and correct one of them.

Problem: Let S be the set of all real numbers x such that $x \neq -2$. Describe the set of all elements $x \in S$ such that

$$\frac{12}{x+2} < 4. \tag{3.23}$$

Note that if $x \in S$ then $\frac{12}{x+2}$ is defined.

ARGUMENT A: Let x be an arbitrary element of S. Then

$$\begin{aligned} \frac{12}{x+2} < 4 &\iff 12 < 4x + 8 \\ &\iff 0 < 4x - 4 \\ &\iff 0 < 4(x-1) \\ &\iff 0 < x - 1 \\ &\iff 1 < x. \end{aligned}$$

Hence the set of all real numbers that satisfy inequality (3.23) is the set of all real numbers x such that $1 < x$. ‖

ARGUMENT B: Let x be an arbitrary element of S. Then

$$\begin{aligned} \frac{12}{x+2} < 4 &\implies 0 < 4 - \frac{12}{x+2} \\ &\implies 0 < \frac{4x + 8 - 12}{x+2} \end{aligned}$$

$$\begin{aligned}
&\Longrightarrow \quad 0 < \frac{4x-4}{x+2} \\
&\Longrightarrow \quad 0 < \frac{4(x-1)}{x+2} \\
&\Longrightarrow \quad 0 < \frac{x-1}{x+2}.
\end{aligned}$$

Now

$$\begin{aligned}
0 < \frac{x-1}{x+2} \quad &\Longleftrightarrow \quad (0 < x-1 \text{ and } \ 0 < x+2) \text{ or } \ (0 > x-1 \text{ and } \ 0 > x+2) \\
&\Longleftrightarrow \quad (1 < x \text{ and } \ -2 < x) \text{ or } \ (1 > x \text{ and } \ -2 > x) \\
&\Longleftrightarrow \quad 1 < x \text{ or } \ -2 > x.
\end{aligned}$$

Hence the set of all real numbers that satisfy inequality (3.23) is the set of all $x \in \mathbf{R}$ such that either $x < -2$ or $x > 1$. $\|$

3.2 Sets Defined by Propositions

The most common way of describing sets is by means of proposition forms.

3.24 Notation ($\{x : P(x)\}$) Let P be a proposition form over a set S, and let T be a subset of S. Then

$$\{x : x \in T \text{ and } \ P(x)\} \tag{3.25}$$

is defined to be the set of all elements x in T such that $P(x)$ is true. The set described in (3.25) is also written

$$\{x \in T : P(x)\}.$$

In cases where the meaning of "T" is clear from the context, we may abbreviate (3.25) by

$$\{x : P(x)\}.$$

3.26 Examples.

$$\{x \in \mathbf{Z} \ : \ \text{for some } y \in \mathbf{Z} \ (x = 2y)\}$$

is the set of all even integers, and

$$\mathbf{Z}^+ \ = \{x : x \in \mathbf{Z} \ \text{ and } \ x > 0\}.$$

If A and B are sets, then

$$A \cup B = \{x : x \in A \text{ or } x \in B\}, \tag{3.27}$$
$$A \cap B = \{x : x \in A \text{ and } x \in B\}, \tag{3.28}$$
$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}. \tag{3.29}$$

We will use the following notation throughout these notes.

3.30 Notation ($\mathbf{Z}_{\geq n}$, $\mathbf{R}_{\geq a}$) If n is an integer we define

$$\mathbf{Z}_{\geq n} = \{k \in \mathbf{Z} : k \geq n\}.$$

Thus

$$\mathbf{Z}_{\geq 1} = \mathbf{Z}^+ \text{ and } \mathbf{Z}_{\geq 0} = \text{the set of non-negative integers} = \mathbf{N}.$$

Similarly, if a is a real number, we define

$$\mathbf{R}_{\geq a} = \{x \in \mathbf{R} : x \geq a\}.$$

3.31 Definition (Ordered pair.) If a, b are objects, then the *ordered pair* (a, b) is a new object obtained by combining a and b. Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$. Similarly we may consider *ordered triples*. Two ordered triples (a, b, x) and (c, d, y) are equal if and only if $a = c$ and $b = d$ and $x = y$. We use the same notation (a, b) to represent an open interval in $\mathbf{R}$ and an ordered pair in $\mathbf{R}^2$. The context should always make it clear which meaning is intended.

3.32 Definition (Cartesian product) If A, B are sets then the *Cartesian product of A and B* is defined to be the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$:

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\} \tag{3.33}$$

3.34 Examples. Let a, b, c, d be real numbers with $a \leq b$ and $c \leq d$. Then

$$[a, b] \times [c, d] = B(a, b : c, d)$$

and

$$[c, d] \times [a, b] = B(c, d : a, b).$$

Thus in general $A \times B \neq B \times A$.

The set $A \times A$ is denoted by A^2. You are familiar with one Cartesian product. The euclidean plane $\mathbf{R}^2$ is the Cartesian product of $\mathbf{R}$ with itself.

3.35 Exercise. Let $S = B(-2, 2 : -2, 2)$ and let
$R_1 = \{(x, y) \in S : xy \leq 0\}$
$R_2 = \{(x, y) \in S : x^2 - 1 \leq 0\}$
$R_3 = \{(x, y) \in S : y^2 - 1 \leq 0\}$
$R_4 = \{(x, y) \in S : xy(x^2 - 1)(y^2 - 1) \leq 0\}$
Sketch the sets S, R_1, R_2, R_3, R_4. For R_4 you should include an explanation of how you arrived at your answer. For the other sets no explanation is required.

3.36 Exercise. Do there exist sets A, B such that $A \times B$ has exactly five elements?

3.3 Functions

3.37 Definition (Function.) Let A, B be sets. A *function with domain A and codomain B* is an ordered triple (A, B, f), where f is a rule which assigns to each element of A a unique element of B. The element of B which f assigns to an element x of A is denoted by $f(x)$. We call $f(x)$ the *f-image of x* or the *image of x under f*. The notation $f : A \longrightarrow B$ is an abbreviation for "f is a function with domain A and codomain B". We read "$f : A \longrightarrow B$" as "f is a function from A to B."

3.38 Examples. Let $f : \mathbf{Z} \longrightarrow \mathbf{N}$ be defined by the rule

$$f(n) = n^2 \text{ for all } n \in \mathbf{Z}.$$

Then $f(2) = 4, f(-2) = 4,$ and $f(1/2)$ is not defined, because $1/2 \notin \mathbf{Z}$.

Let $g : \mathbf{N} \longrightarrow \mathbf{N}$ be defined by the rule: for all $n \in \mathbf{N}$

$$g(n) = \text{the last digit in the decimal expansion for } n.$$

Thus $g(21) = 1$, $g(0) = 0$, $g(1984) = 4$, $g(666) = 6$.

3.39 Definition (Maximum and minimum functions.) We define functions max and min from $\mathbf{R}^2$ to $\mathbf{R}$ by the rule

$$\max(x, y) = \begin{cases} x & \text{if } x \geq y \\ y & \text{otherwise.} \end{cases} \tag{3.40}$$

$$\min(x,y) = \begin{cases} y & \text{if } x \geq y \\ x & \text{otherwise.} \end{cases} \tag{3.41}$$

Thus we have

$$\min(x,y) \leq x \leq \max(x,y)$$

and

$$\min(x,y) \leq y \leq \max(x,y)$$

for all $(x,y) \in \mathbf{R}^2$. Also

$$\max(2,7) = 7 \text{ and } \min(-2,-7) = -7.$$

3.42 Definition (Absolute value function.) Let $A\colon \mathbf{R} \to \mathbf{R}$ be defined by the rule

$$A(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0. \end{cases}$$

We call A the *absolute value function* and we usually designate $A(x)$ by $|x|$.

3.43 Definition (Sequence) Let S be a set. A *sequence* in S is a function $f\colon \mathbf{Z}^+ \to S$. I will refer to a sequence in $\mathbf{R}$ as a *real sequence*.

The sequence f is sometimes denoted by $\{f(n)\}$. Thus $\left\{\frac{1}{n^2+1}\right\}$ is the sequence $f\colon \mathbf{Z}^+ \to \mathbf{R}$ such that $f(n) = \frac{1}{n^2+1}$ for all $n \in \mathbf{Z}^+$. Sometimes the sequence f is denoted by

$$\{f(1), f(2), f(3), \cdots\}, \tag{3.44}$$

for example $\{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is the same as $\{\frac{1}{n}\}$. The notation in formula (3.44) is always ambiguous. I will use it for sequences like

$$\{1, 1, -1, -1, 1, 1, -1, -1, 1, 1\cdots\}$$

in which it is somewhat complicated to give an analytic description for $f(n)$.

If f is a sequence, and $n \in \mathbf{Z}^+$, then we often denote $f(n)$ by f_n.

3.45 Examples. Let P denote the set of all polygons in the plane. For each number a in $\mathbf{R}^+$ let

$$S_a^2 = \{(x,y) \in \mathbf{R}^2 : 0 \leq x \leq a \text{ and } 0 \leq y \leq x^2\}.$$

For each $n \in \mathbf{Z}^+$ let

$$Q_n = \bigcup_{i=1}^{n} I_i$$

and

$$R_n = \bigcup_{i=1}^{n} O_i$$

denote the polygons inscribed in S_a^2 and containing S_a^2 described on page 20. Then

$\{Q_n\}$ and $\{R_n\}$ are sequences in P.

$\{\text{area}(Q_n)\} = \{\frac{a^3}{3}(1-\frac{1}{n})(1-\frac{1}{2n})\}$ is a real sequence. (Cf. (2.3) and (2.12).)

$\{[\text{area}(Q_n), \text{area}(R_n)]\}$ is a sequence of intervals.

3.46 Definition (Equality for functions.) Let (A, B, f) and (C, D, g) be two functions. Then, since a function is an ordered triple, we have

$$(A, B, f) = (C, D, g) \text{ if and only if } A = C \text{ and } B = D, \text{ and } f = g.$$

The rules f and g are equal if and only if $f(a) = g(a)$ for all $a \in A$. If $f : A \longrightarrow B$ and $g : C \longrightarrow D$ then it is customary to write $f = g$ to mean $(A, B, f) = (C, D, g)$. This is an abuse of notation, but it is a standard practice.

3.47 Examples. If $f : \mathbf{Z} \longrightarrow \mathbf{Z}$ is defined by the rule

$$f(x) = x^2 \text{ for all } x \text{ in } \mathbf{Z}$$

and $g : \mathbf{Z} \longrightarrow \mathbf{N}$ is defined by the rule

$$g(x) = x^2 \text{ for all } x \text{ in } \mathbf{Z}$$

then $f \neq g$ since f and g have different codomains.

If $f : \mathbf{Q} \longrightarrow \mathbf{Q}$ and $g : \mathbf{Q} \longrightarrow \mathbf{Q}$ are defined by the rules

$$f(x) = x^2 - 1 \text{ for all } x \in \mathbf{Q}$$

$$g(y) = (y - 1)(y + 1) \text{ for all } y \in \mathbf{Q}$$

then $f = g$.

In certain applications it is important to know the precise codomain of a function, but in many applications the precise codomain is not important, and in such cases I will often omit all mention of the codomain. For example, I might say "For each positive number a, let $J(a) = [0, a]$." and proceed as though I had defined a function. Here you could reasonably take the codomain to be the set of real intervals, or the set of closed intervals, or the set of all subsets of $\mathbf{R}$.

3.48 Definition (Image of f) Let A, B be sets, and let $f : A \longrightarrow B$. The set

$$\{y \in B : \text{ for some } x \in A \ (y = f(x))\}$$

is called the *image of* f, and is denoted by $f(A)$. More generally, if T is any subset of A then we define

$$f(T) = \{y \in B : \text{ for some } x \in T \ (y = f(x))\}.$$

We call $f(T)$ the f-image of T. Clearly, for every subset T of A we have $f(T) \subset B$.

3.49 Examples. If $f : \mathbf{Z} \longrightarrow \mathbf{Z}$ is defined by the rule

$$f(n) = n + 3 \text{ for all } n \in \mathbf{Z}$$

then $f(2) = 5$ so $f(2) \in \mathbf{Z}$,
$f(\{2\}) = \{5\}$ so $f(\{2\}) \subset \mathbf{Z}$,
$f(\mathbf{N}) = \mathbf{Z}_{\geq 3}$.

3.50 Definition (Graph of f) Let A, B be sets, and let $f : A \longrightarrow B$. The *graph of* f is defined to be

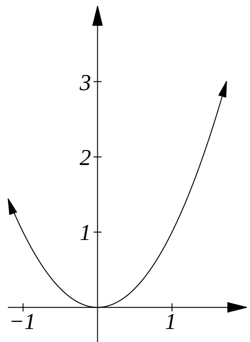
$$\{(x, y) \in A \times B : y = f(x)\}$$

If the domain and codomain of f are subsets of $\mathbf{R}$, then the graph of f can be identified with a subset of the plane.

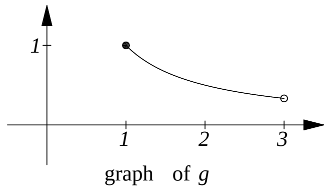
3.51 Examples. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by the rule

$$f(x) = x^2 \text{ for all } x \in \mathbf{R}.$$

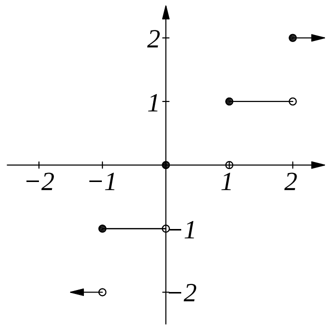
The graph of f is sketched below. The arrowheads on the graph are intended to indicate that the complete graph has not been drawn.



graph of f



graph of g



graph of h

Let $S = \{x \in \mathbf{R} \ : 1 \leq x < 3\}$. Let g be the function from S to $\mathbf{R}$ defined by the rule

$$g(x) = \frac{1}{x} \text{ for all } x \in S.$$

The graph of g is sketched above. The solid dot at $(1, 1)$ indicates that $(1, 1)$ is in the graph. The hollow dot at $(3, 1/3)$ indicates that $(3, 1/3)$ is not in the graph.

Let $h : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by the rule

$$h(x) = \text{ the greatest integer less than or equal to } x.$$

Thus $h(3.14) = 3$ and $h(-3.14) = -4$. The graph of h is sketched above.

The term function (*functio*) was introduced into mathematics by Leibniz [33, page 272 footnote]. During the seventeenth century the ideas of function and curve were usually thought of as being the same, and a curve was often thought of as the path of a moving point. By the eighteenth century the idea of function was associated with "analytic expression". Leonard Euler (1707–1783) gave the following definition:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.
>
> Hence every analytic expression, in which all component quantities except the variable z are constants, will be a function of that

> z; Thus $a + 3z$; $az - 4z^2$; $az + b\sqrt{a^2 - z^2}$; c^z; etc. are functions of z[18, page 3].

The use of the notation "$f(x)$" to represent the value of f at x was introduced by Euler in 1734 [29, page 340].

3.52 Exercise. Sketch the graphs of the following functions:

a) $f(x) = (x - 1)^2$ for all $x \in [0, 4]$.

b) $g(x) = (x - 2)^2$ for all $x \in [-1, 3]$.

c) $h(x) = x^2 - 1$ for all $x \in [-2, 2]$.

d) $k(x) = x^2 - 2^2$ for all $x \in [-2, 2]$.

3.4 Summation Notation

Let k and n be integers with $k \leq n$. Let $x_k, x_{k+1}, \ldots x_n$, be real numbers, indexed by the integers from k to n. We define

$$\sum_{i=k}^{n} x_i = x_k + x_{k+1} + \cdots + x_n, \tag{3.53}$$

i.e. $\sum\limits_{i=k}^{n} x_i$ is the sum of all the numbers $x_k, \ldots x_n$. A sum of one number is defined to be that number, so that

$$\sum_{i=k}^{k} x_i = x_k.$$

The "i" in equation (3.53) is a dummy variable, and can be replaced by any symbol that has no meaning assigned to it. Thus

$$\sum_{j=2}^{4} \frac{1}{j} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}.$$

The following properties of the summation notation should be clear from the definition. (Here $c \in \mathbf{R}$, k and n are integers with $k \leq n$ and $x_k, \ldots, x_n$ and $y_k, \ldots y_n$ are real numbers.)

$$\begin{aligned}
\sum_{j=k}^{n} x_j + \sum_{j=k}^{n} y_j &= \sum_{j=k}^{n} (x_j + y_j). \\
c \sum_{j=k}^{n} x_j &= \sum_{j=k}^{n} c x_j. \\
\sum_{j=k}^{n} 1 &= \sum_{j=k}^{n} 1^j = n - k + 1. \\
\left(\sum_{j=k}^{n} x_j \right) + x_{n+1} &= \sum_{j=k}^{n+1} x_j.
\end{aligned}$$

If $x_j \leq y_j$ for all j satisfying $k \leq j \leq n$ then

$$\sum_{j=k}^{n} x_j \leq \sum_{j=k}^{n} y_j.$$

Also

$$\sum_{j=k}^{n} x_j = \sum_{j=k-1}^{n-1} x_{j+1} = \sum_{j=k+1}^{n+1} x_{j-1} = x_k + \cdots + x_n.$$

Using the summation notation, we can rewrite equations (2.9) and (2.23) as

$$\sum_{p=1}^{n} p^2 = \frac{n(n+1)(2n+1)}{6}$$

and

$$\sum_{j=0}^{n-1} r^j = \frac{1 - r^n}{1 - r}.$$

The use of the Greek letter Σ to denote sums was introduced by Euler in 1755[15, page 61]. Euler writes

$$\Sigma x^2 = \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6}.$$

Compare this with the notation in Bernoulli's table on page 27. (The apparent difference is due to the fact that for Euler, Σx^2 denotes the sum of x squares, starting with 0^2, whereas for Bernoulli $\int nn$ denotes the sum of n squares starting with 1^2.) The use of the symbol $\int$ (which is a form of S) for sums was introduced by Leibniz. The use of limits on sums was introduced by Augustin Cauchy(1789-1857). Cauchy used the notation $\sum_{m}^{n}{}_r\, fr$ to denote what we would write as $\sum_{r=m}^{n} f(r)$[15, page 61].

3.54 Exercise. Find the following sums:

a) $\sum_{j=1}^{n}(2j-1)$ for $n = 1, 2, 3, 4$.

b) $\sum_{j=1}^{n} \frac{1}{j(j+1)}$ for $n = 1, 2, 3, 4$.

c) $\sum_{j=1}^{9} \frac{9}{10^j}$.

3.5 Mathematical Induction

The induction principle is a way of formalizing the intuitive idea that if you begin at 1 and start counting "$1, 2, 3, \ldots$", then eventually you will reach any preassigned number (such as for example, 200004).

3.55 Assumption (The Induction Principle) Let k be an integer, and let P be a proposition form over $\mathbf{Z}_{\geq k}$. If

$P(k)$ is true,

and

"for all $n \in \mathbf{Z}_{\geq k}[P(n) \Longrightarrow P(n+1)]$" is true,

then

"for all $n \in \mathbf{Z}_{\geq k}[P(n)]$" is true.

In order to prove "for all $n \in \mathbf{Z}_{\geq k}, P(n)$" by using the induction principle, you should

1. Prove that $P(k)$ is true.
2. Take a generic element n of $\mathbf{Z}_{\geq k}$ and prove $(P(n) \Longrightarrow P(n+1))$.

Recall that the way to prove "$P(n) \Longrightarrow P(n+1)$" is true, is to assume that $P(n)$ is true and show that then $P(n+1)$ must be true.

3.56 Example. We will use the induction principle to do exercise 2.10. For all $n \in \mathbf{Z}_{\geq 1}$, let

$$P(n) = \left[\sum_{p=1}^{n} p^3 = \frac{n^2(n+1)^2}{4}\right].$$

Then $P(1)$ says

$$\sum_{p=1}^{1} p^3 = \frac{(1^2)(1+1)^2}{4}$$

which is true, since both sides of this equation are equal to 1. Now let n be a generic element of $\mathbf{Z}_{\geq 1}$ Then

$$\begin{aligned}
P(n) \quad &\Longleftrightarrow \quad \sum_{p=1}^{n} p^3 = \frac{n^2(n+1)^2}{4} \\
&\Longrightarrow \quad \left(\sum_{p=1}^{n} p^3\right) + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&\Longrightarrow \quad \sum_{p=1}^{n+1} p^3 = \frac{(n+1)^2}{4}(n^2 + 4(n+1)) = \frac{(n+1)^2}{4}(n^2 + 4n + 4) \\
&\Longrightarrow \quad \sum_{p=1}^{n+1} p^3 = \frac{(n+1)^2(n+2)^2}{4} \\
&\Longleftrightarrow \quad P(n+1).
\end{aligned}$$

It follows from the induction principle that $P(n)$ is true for all $n \in \mathbf{Z}_{\geq 1}$, which is what we wanted to prove. |||

3.57 Example. We will show that for all $n \in \mathbf{Z}_{\geq 4}[n! > 2^n]$.
Proof: Define a proposition form P over $\mathbf{Z}_{\geq 4}$ by

$$P(n) = [n! > 2^n].$$

Now $4! = 24 > 16 = 2^4$, so $4! > 2^4$ and thus $P(4)$ is true.

Let n be a generic element of $\mathbf{Z}_{\geq 4}$. Since $n \in \mathbf{Z}_{\geq 4}$, we know that

$$n + 1 \geq 4 + 1 > 2.$$

Hence

$$\begin{aligned} P(n) &\Longrightarrow (n! > 2^n > 0) \text{ and } (n + 1 > 2 > 0) \\ &\Longrightarrow (n + 1) \cdot n! > 2 \cdot 2^n \\ &\Longrightarrow ((n + 1)! > 2^{n+1}) \Longrightarrow P(n + 1). \end{aligned}$$

Hence, for all $n \in \mathbf{Z}_{\geq 4}[P(n) \Longrightarrow P(n + 1)]$. It follows from the induction principle that for all $n \in \mathbf{Z}_{\geq 4}[n! > 2^n]$. $\|$