Chapter 0

Introduction

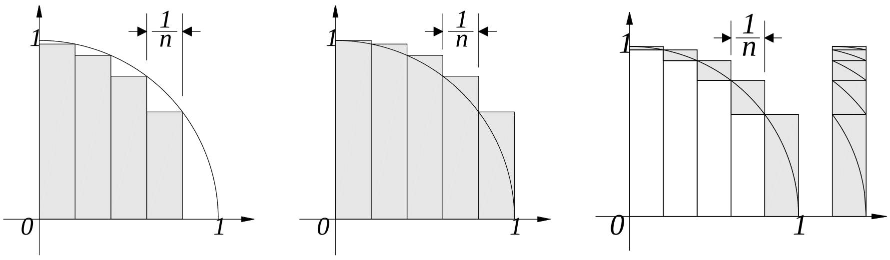
An Overview of the Course

In the first part of these notes we consider the problem of calculating the areas of various plane figures. The technique we use for finding the area of a figure A will be to construct a sequence I_n of sets contained in A, and a sequence O_n of sets containing A, such that

1. The areas of I_n and O_n are easy to calculate.

2. When n is large then both I_n and O_n are in some sense "good approximations" for A.

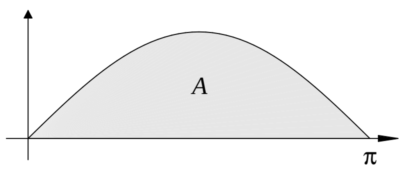
Then by examining the areas of I_n and O_n we will determine the area of A. The figure below shows the sorts of sets we might take for I_n and O_n in the case where A is the set of points in the first quadrant inside of the circle $x^2+y^2 = 1$.



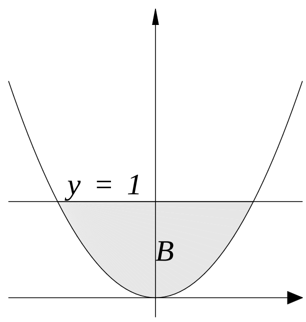
I_n $\quad$ O_n $\quad$ $\text{area}(O_n) - \text{area}(I_n) = \frac{1}{n}$

In this example, both of the sets I_n and O_n are composed of a finite number of rectangles of width $\frac{1}{n}$, and from the equation of the circle we can calculate the heights of the rectangles, and hence we can find the areas of I_n and O_n. From the third figure we see that area$(O_n)-$ area$(I_n) = \frac{1}{n}$. Hence if $n = 100000$, then either of the numbers area(I_n) or area(O_n) will give the area of the quarter-circle with an error of no more than 10^{-5}. This calculation will involve taking many square roots, so you probably would not want to carry it out by hand, but with the help of a computer you could easily find the area of the circle to five decimals accuracy. However no amount of computing power would allow you to get thirty decimals of accuracy from this method in a lifetime, and we will need to develop some theory to get better approximations.

In some cases we can find exact areas. For example, we will show that the area of one arch of a sine curve is 2, and the area bounded by the parabola $y = x^2$ and the line $y = 1$ is $\frac{4}{3}$.



$y = \sin(x)$ area$(A) = 2$



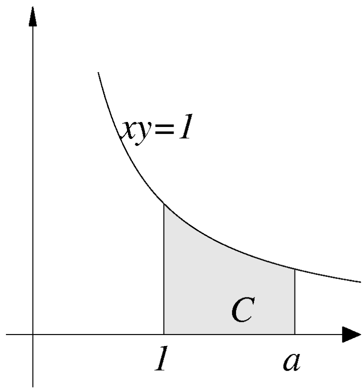
$y = x^2$ area$(B) = \frac{4}{3}$

However in other cases the areas are not simply expressible in terms of known numbers. In these cases we define certain numbers in terms of areas, for example we will define

$$\pi = \text{the area of a circle of radius } 1,$$

and for all numbers $a > 1$ we will define

$$\begin{aligned} \ln(a) &= \text{the area of the region bounded by the curves} \\ &\quad\; y = 0,\ xy = 1,\ x = 1, \text{ and } x = a. \end{aligned}$$



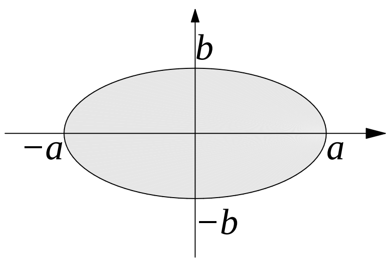
area(C) = ln(a)

We will describe methods for calculating these numbers to any degree of accuracy, and then we will consider them to be known numbers, just as you probably now think of $\sqrt{2}$ as being a known number. (Many calculators calculate these numbers almost as easily as they calculate square roots.) The numbers $\ln(a)$ have many interesting properties which we will discuss, and they have many applications to mathematics and science.

Often we consider general classes of figures, in which case we want to find a simple formula giving areas for all of the figures in the class. For example we will express the area of the ellipse bounded by the curve whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

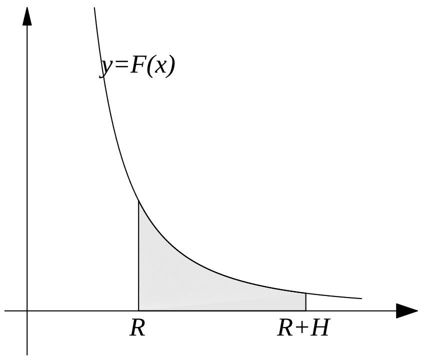
by means of a simple formula involving a and b.



The mathematical tools that we develop for calculating areas, (i.e. the theory of *integration*) have many applications that seem to have little to do with area. Consider a moving object that is acted upon by a known force $F(x)$ that depends on the position x of the object. (For example, a rocket propelled upward from the surface of the moon is acted upon by the moon's gravitational attraction, which is given by

$$F(x) = \frac{C}{x^2},$$

where x is the distance from the rocket to the center of the moon, and C is some constant that can be calculated in terms of the mass of the rocket and known information.) Then the amount of work needed to move the object from a position $x = x_0$ to a position $x = x_1$ is equal to the area of the region bounded by the lines $x = x_0$, $x = x_1$, $y = 0$ and $y = F(x)$.



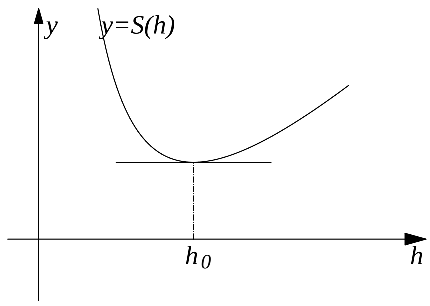
Work is represened by an area

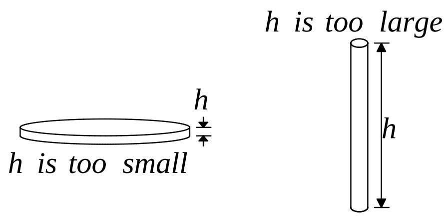
In the case of the moon rocket, the work needed to raise the rocket a height H above the surface of the moon is the area bounded by the lines $x = R$, $x = R + H$, $y = 0$, and $y = \frac{C}{x^2}$, where R is the radius of the moon. After we have developed a little bit of machinery, this will be an easy area to calculate. The amount of work here determines the amount of fuel necessary to raise the rocket.

Some of the ideas used in the theory of integration are thousands of years old. Quite a few of the technical results in the calculations presented in these notes can be found in the writings of Archimedes(287–212 B.C.), although the way the ideas are presented here is not at all like the way they are presented by Archimedes.

In the second part of the notes we study the idea of *rate of change*. The ideas used in this section began to become common in early seventeenth century, and they have no counterpart in Greek mathematics or physics. The problems considered involve describing motions of moving objects (e.g. cannon balls or planets), or finding tangents to curves. An important example of a rate of change is *velocity*. The problem of what is meant by the velocity of a moving object at a given instant is a delicate one. At a particular instant of time, the object occupies just one position in space. Hence during that instant the object does not move. If it does not move, it is at rest. If it is at rest, then its velocity must be 0(?)

The ability to find tangents to curves allows us to find maximum and minimum values of functions. Suppose I want to design a tin can that holds 1000 cc., and requires a minimum amount of tin. It is not hard to find a function S such that for each positive number h, the total surface area of a can with height h and volume 1000 is equal to $S(h)$. The graph of S has the general shape shown in the figure, and the minimum surface area corresponds to the height h_0 shown in the figure. This value h_0 corresponds to the point on the graph of S where the tangent line is horizontal, i.e. where the slope of the tangent is zero. From the formula for $S(h)$ we will be able to find a formula for the slope of the tangent to the graph of S at an arbitrary height h, and to determine when the slope is zero. Thus we will find h_0.





Badly designed cans

The tool for solving rate problems is the *derivative*, and the process of calculating derivatives is called *differentiation*. (There are two systems of notation working here. The term *differential* was introduced by Gottfried Leibniz(1646–1716) to describe a concept that later developed into what Joseph Louis Lagrange(1736–1813) called the *derived function*. From Lagrange we get our word *derivative*, but the older name due to Leibniz is still used to describe the general theory – from which differentials in the sense of Leibniz have been banished.) The idea of derivative (or fluxion or differential) appears in the work of Isaac Newton(1642–1727) and of Leibniz, but can be found in various disguises in the work of a number of earlier mathematicians.

As a rule, it is quite easy to calculate the velocity and acceleration of a moving object, if a formula for the position of the object at an arbitrary time is known. However usually no such formula is obvious. Newton's Second Law states that the acceleration of a moving object is proportional to the sum of the forces acting on the object, divided by the mass of the object. Now often we have a good idea of what the forces acting on an object are, so we know the acceleration. The interesting problems involve calculating velocity and position from acceleration. This is a harder problem than the problem going in the opposite direction, but we will find ways of solving this problem in many cases. The natural statements of many physical laws require the notion of derivative for their statements. According to Salomon Bochner

> The mathematical concept of derivative is a master concept, one of the most creative concepts in analysis and also in human cognition altogether. Without it there would be no velocity or acceleration or momentum, no density of mass or electric charge or any other density, no gradient of a potential and hence no concept of potential in any part of physics, no wave equation; no mechanics no physics, no technology, nothing[11, page 276].

At the time that ideas associated with differentiation were being developed, it was widely recognized that a logical justification for the subject was completely lacking. However it was generally agreed that the results of the calculations based on differentiation were correct. It took more than a century before a logical basis for derivatives was developed, and the concepts of *function* and *real number* and *limit* and *continuity* had to be developed before the foundations could be described. The story is probably not complete. The modern "constructions" of real numbers based on a general theory of "sets" appear to me to be very vague, and more closely related to philosophy than to mathematics. However in these notes we will not worry about the foundations of the real numbers. We will assume that they are there waiting for us to use, but we will need to discuss the concepts of function, limit and continuity in order to get our results.

The *fundamental theorem of the calculus* says that the theory of integration, and the theory of differentiation are very closely related, and that differentiation techniques can be used for solving integration problems, and vice versa. The fundamental theorem is usually credited to Newton and Leibniz independently, but it can be found in various degrees of generality in a number of earlier writers. It was an idea floating in the air, waiting to be discovered at the close of the seventeenth century.

Prerequisites

The prerequisites for this course are listed in appendix C. You should look over this appendix, and make sure that everything in it is more or less familiar to you. If you are unfamiliar with much of this material, you might want to discuss with your instructor whether you are prepared to take the course. It will be helpful to have studied some trigonometry, but all of the trigonometry used in these notes will be developed as it is needed.

You should read these notes carefully and critically. There are quite a few cases where I have tried to trick you by giving proofs that use unjustified assumptions. In these cases I point out that there is an error after the proof is complete, and either give a new proof, or add some hypotheses to the statement of the theorem. If there is something in a proof that you do not understand, there is a good chance that the proof is wrong.

Exercises and Entertainments

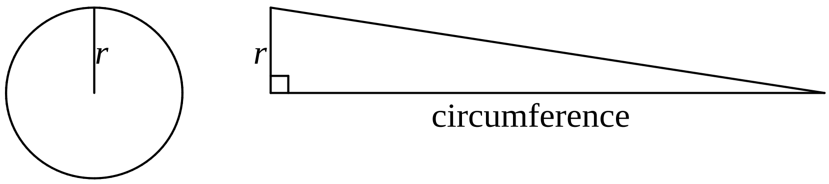
The exercises are an important part of the course. Do not expect to be able to do all of them the first time you try them, but you should understand them after they have been discussed in class. Some important theorems will be proved in the exercises. There are hints for some of the questions in appendix A, but you should not look for a hint unless you have made some effort to answer a question.

Sections whose titles are marked by an asterisk (e.g. section 2.6) are not used later in the notes, and may be omitted. Hovever they contain really neat material, so you will not want to omit them.

In addition to the exercises, there are some questions and statements with the label "entertainment". These are for people who find them entertaining. They require more time and thought than the exercises. Some of them are more metaphysical than mathematical, and some of them require the use of a computer or a programmable calculator. If you do not find the entertainments entertaining, you may ignore them. Here is one to start you off.

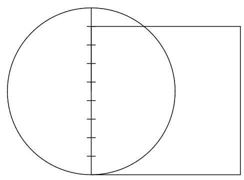
1 Entertainment (Calculation of π.) . The area of a circle of radius 1 is denoted by π. Calculate π as accurately as you can.

Archimedes showed that π is half of the circumference of a circle of radius 1. More precisely, he showed that the area of a circle is equal to the area of a triangle whose base is equal to the circumference of the circle, and whose altitude is equal to the radius of the circle. If we take a circle of radius 1, we get the result stated.



You should assume Archimedes' theorem, and then entertainment 1 is equivalent to the problem of calculating the circumference of a circle as accurately as you can. An answer to this problem will be a pair of rational numbers b and c, together with an argument that $b < \pi$ and $\pi < c$. It is desired to make the difference $c - b$ as small as possible.

This problem is very old. The Rhind Papyrus[16, page 92] (c. 1800 B.C.?) contains the following rule for finding the area of a circle:



RULE I: Divide the diameter of the circle into nine equal parts, and form a square whose side is equal to eight of the parts. Then the area of the square is equal to the area of the circle.

The early Babylonians (1800-1600BC) [38, pages 47 and 51] gave the following rule:

RULE II: The area of a circle is 5/60th of the square of the circumference of the circle.

Archimedes (287–212 B.C.) proved that the circumference of a circle is three times the diameter plus a part smaller than one seventh of the diameter, but greater than 10/71 of the diameter[3, page 134]. In fact, by using only elementary geometry, he gave a method by which π can be calculated to any degree of accuracy by someone who can calculate square roots to any degree of accuracy. We do not know how Archimedes calculated square roots, but people have tried to figure out what method he used by the form of his approximations. For example he says with no justification that

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}$$

and

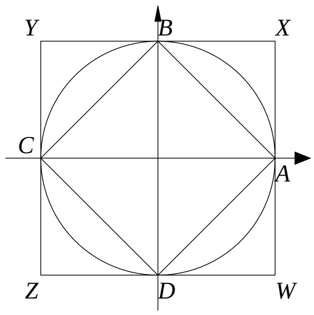
$$\sqrt{3380929} < 1838\frac{9}{11}.$$

By using your calculator you can easily verify that these results are correct. Presumably when you calculate π you will use a calculator or computer to estimate any square roots you need. This immediately suggests a new problem.

2 Entertainment (Square root problem.) Write, or at least describe, a computer program that will calculate square roots to a good deal of accuracy. This program should use only the standard arithmetic operations and the constructions available in all computer languages, and should not use any special functions like square roots or logarithms. An answer to this question must include some sort of explanation of why the method works.

Zǔ Chōngzhī (429–500 A.D.) stated that π is between 3.1415926 and 3.1415927, and gave 355/113 as a good approximation to π.[47, page 82]

Here is a first approximation to π. Consider a circle of radius 1 with center at $(0, 0)$, and inscribe inside of it a square $ABCD$ of side s with vertices at $(1, 0), (0, 1), (-1, 0)$ and $(0, -1)$. Then by the Pythagorean theorem, $s^2 = 1^2 + 1^2 = 2$. But s^2 is the area of the square $ABCD$, and since $ABCD$ is contained inside of the circle we have



$$2 = \text{Area of inscribed square} < \text{Area of circle} = \pi.$$

Consider also the circumscribed square $WXYZ$ with horizontal and vertical sides. This square has side 2, and hence has area 4. Thus, since the circle is contained in $WXYZ$,

$$\pi = \text{area of circle} < \text{area}(WXYZ) = 4.$$

It now follows that $2 < \pi < 4$.

A number of extraordinary formulas for π are given in a recent paper on *How to Compute One Billion Digits of Pi*[12]. One amazing formula given in this paper is the following result

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \frac{[1103 + 26390n]}{396^{4n}},$$

which is due to S. Ramanujan(1887–1920)[12, p 201,p 215]. The reciprocal of the zeroth term of this sum i.e.

$$\frac{9801}{1103\sqrt{8}}$$

gives a good approximation to π (see exercise 4).

3 Exercise. The formulas described in RULES I and II above each determine an approximate value for π. Determine the two approximate values. Explain your reasoning.

4 Exercise. Use a calculator to find the value of

$$\frac{9801}{1103\sqrt{8}},$$

and compare this with the correct value of π, which is $3.14159265358979\ldots$.