Appendix B

Proofs of Some Area Theorems

B.1 Theorem (Addition Theorem.) *For any bounded sets S and T in* $\mathbf{R}^2$

$$\alpha(S \cup T) = \alpha(S) + \alpha(T) - \alpha(S \cap T). \tag{B.2}$$

and consequently

$$\alpha(S \cup T) \leq \alpha(S) + \alpha(T).$$

Proof: We have

$$S \cup T = S \cup (T \setminus S) \text{ where } S \cap (T \setminus S) = \emptyset$$

and

$$T = (T \setminus S) \cup (T \cap S) \text{ where } (T \setminus S) \cap (T \cap S) = \emptyset.$$

Hence by the additivity of area

$$\alpha(S \cup T) = \alpha(S) + \alpha(T \setminus S) \tag{B.3}$$

and

$$\alpha(T) = \alpha(T \setminus S) + \alpha(T \cap S) \tag{B.4}$$

If we solve equation (B.4) for $\alpha(T \setminus S)$ and use this result in equation (B.3) we get the desired result. ‖

B.5 Corollary (Subadditivity of area.) *Let $n \in \mathbf{Z}_{\geq 1}$, and let A_1, A_2, $\cdots$ A_n be bounded sets in $\mathbf{R}^2$. Then*

$$\alpha(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \alpha(A_i). \tag{B.6}$$

Proof: The proof is by induction. If $n = 1$, then (B.6) says $\alpha(A_1) \leq \alpha(A_1)$, which is true. Suppose now that k is a generic element of $\mathbf{Z}_{\geq 1}$, and that (B.6) is true when $n = k$. Let $A_1, \cdots, A_{k+1}$ be bounded sets in $\mathbf{R}^2$. Then

$$\begin{aligned}
\alpha(\bigcup_{i=1}^{k+1} A_i) &= \alpha\left(\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right) \\
&\leq \alpha\left(\bigcup_{i=1}^{k} A_i\right) + \alpha(A_{k+1}) \\
&\leq \sum_{i=1}^{k} \alpha(A_i) + \alpha(A_{k+1}) = \sum_{i=1}^{k+1} \alpha(A_i).
\end{aligned}$$

Hence (B.6) is true when $n = k + 1$, and by induction the formula holds for all $n \in \mathbf{Z}_{n \geq k}$. ‖

B.7 Theorem (Monotonicity of Area.) *Let S, T be bounded sets such that $S \subset T$. Then $\alpha(S) \leq \alpha(T)$.*

Proof: If $S \subset T$ then $S \cap T = S$, and in this case equation (B.4) becomes

$$\alpha(T) = \alpha(T \setminus S) + \alpha(S).$$

Since $\alpha(T \setminus S) \geq 0$, it follows that $\alpha(T) \geq \alpha(S)$. ‖

B.8 Theorem (Additivity for almost disjoint sets.) *Let $\{R_1, \cdots, R_n\}$ be a finite set of bounded sets such that R_i and R_j are almost disjoint whenever $i \neq j$. Then*

$$\alpha(\bigcup_{i=1}^{n} R_i) = \sum_{i=1}^{n} \alpha(R_i). \tag{B.9}$$

Proof: The proof is by induction on n. For $n = 1$, equation (B.9) says that $\alpha(R_1) = \alpha(R_1)$, and this is true. Now suppose $\{R_1 \cdots R_{n+1}\}$ is a family of mutually almost-disjoint sets. Then

$$(R_1 \cup \cdots \cup R_n) \cap R_{n+1} = (R_1 \cap R_{n+1}) \cup (R_2 \cap R_{n+1}) \cup \cdots \cup (R_n \cap R_{n+1})$$

and this is a finite union of zero-area sets, and hence is a zero-area set. Hence, by the addition theorem,

$$\alpha((R_1 \cup \cdots \cup R_n) \cup R_{n+1}) = \alpha(R_1 \cup \cdots \cup R_n) + \alpha(R_{n+1})$$

i.e.,

$$\alpha(\bigcup_{i=1}^{n+1} R_i) = \sum_{i=1}^{n} \alpha(R_i) + \alpha(R_{n+1}) = \sum_{i=1}^{n+1} \alpha(R_i).$$

The theorem now follows from the induction principle. ‖