SKETCH OF THE RIEMANN–VON MANGOLDT EXPLICIT FORMULA

Contents

In the online writeup

`http://www.math.umn.edu/~garrett/m/mfms/notes_c/mfms_notes_02.pdf`

Paul Garrett says

> Even more interesting than a Prime Number Theorem is the *precise* relationship between primes and zeros of zeta found by Riemann. The idea applies to any zeta or L-function for which we know an analytic continuation and other reasonable properties.
>
> It took 40 years for [Hadamard 1893], [vonMangoldt 1895], and others to complete Riemann's 1857-8 sketch of the *Explicit Formula* relating primes to zeros of the Euler-Riemann zeta function. Even then, lacking a zero-free strip inside the critical strip, the Explicit Formula does *not* yield a Prime Number Theorem, despite giving a precise relationship between primes and zeros of zeta.
>
> The *idea* is that the equality of the Euler product and Riemann-Hadamard product for zeta allows extraction of an *exact formula* for a suitably-weighted counting of primes, a sum over zeros of zeta, via a contour integration of the logarithmic derivatives. As observed by Weil, the classical formulas are instances of evaluations of a certain *distribution*, in the sense of *generalized functions*.

This writeup repeats Garrett's encapsulation of the idea of the argument, not addressing the difficulties.

1. Product expressions for the zeta function

The Euler–Riemann zeta function has the product expansion

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}, \quad \mathrm{Re}(s) > 1.$$

This expansion shows that ζ has no zeros in its initial right half plane domain. But also, the completed Euler–Rieman zeta function,

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1,$$

has an integral representation that extends meromorphically to all of $\mathbb{C}$,

$$Z(s) = \int_{t=1}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 t}(t^{s/2} + t^{(1-s)/2}) \frac{\mathrm{d}t}{t} - \frac{1}{s} - \frac{1}{1-s}, \quad s \in \mathbb{C},$$

and the extended Z visibly satisfies the functional equation

$$Z(1-s) = Z(s), \quad s \in \mathbb{C}.$$

Because also $\Gamma(s)$ has a similar representation

$$\Gamma(s) = \int_{1}^{\infty} e^{-t} t^s \frac{\mathrm{d}t}{t} + \sum_{n \geq 0} \frac{(-1)^n}{n!(s+n)}, \quad s \in \mathbb{C},$$

and functional equation

$$\Gamma(s)\Gamma(1-s) = \pi / \sin(\pi s), \quad s \in \mathbb{C},$$

it follows that $\zeta(s)$ extends meromorphically to $\mathbb{C}$ as well. Because $Z(s)$ has simple poles at $s = 0, 1$ and is otherwise analytic, and because the gamma function has simple poles at $s = 0, -1, -2, \cdots$, we see that:

> *The extended $\zeta(s)$ has zeros at $s = -2, -4, -6, \cdots$; these are its* **trivial zeros**. *Any other zeros of $\zeta(s)$ lie in the* **critical strip** $0 \leq \mathrm{Re}(s) \leq 1$.

A growth bound of $Z(s)$ shows that the entire function $(s-1)\zeta(s)$ has a Hadamard product expansion (in which the symbol ρ denotes nontrivial zeros of $\zeta(s)$),

$$(s-1)\zeta(s) = e^{a+bs} \prod_{n \geq 1} \left(1 + \frac{s}{2n}\right) e^{-s/2n} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}, \quad s \in \mathbb{C}.$$

Equate the two product expressions for $\zeta(s)$ to get

$$\prod_{p} (1 - p^{-s})^{-1} = \frac{e^{a+bs}}{(s-1)} \prod_{n \geq 1} \left(1 + \frac{s}{2n}\right) e^{-s/2n} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}, \quad \mathrm{Re}(s) > 1.$$

Take logarithmic derivatives to obtain two expressions for $\zeta'(s)/\zeta(s)$,

$$-\sum_{\substack{p \\ m \geq 1}} \log p \cdot p^{-ms} = b - \frac{1}{s-1} - \sum_{n \geq 1} \frac{s}{2n(s+2n)} + \sum_{\rho} \frac{s}{\rho(s-\rho)}, \quad \mathrm{Re}(s) > 1.$$

Because the right side is $\zeta'(s)/\zeta(s)$ for all $s \in \mathbb{C}$, in particular $b + 1 = \zeta'(0)/\zeta(0)$, and thus

$$b - \frac{1}{s-1} = b + 1 - \frac{s}{s-1} = \frac{\zeta'(0)}{\zeta(0)} - \frac{s}{s-1}.$$

In sum so far,

$$(1) \quad \sum_{\substack{p \\ m \geq 1}} \log p \cdot p^{-ms} = \frac{s}{s-1} - \frac{\zeta'(0)}{\zeta(0)} + \sum_{n \geq 1} \frac{s}{2n(s+2n)} - \sum_{\rho} \frac{s}{\rho(s-\rho)}, \quad \mathrm{Re}(s) > 1.$$

2. A useful extraction integral

The plan is to extract number-theoretic information from the previous display (1) via contour integration. To avoid interrupting the story later, we now introduce the identity

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^s}{s}\,\mathrm{d}s=\begin{cases}1 & \text{if } x>1,\\ 0 & \text{if } 0<x<1,\end{cases}\qquad \sigma>0.$$

The idea of the proof is that if $x>1$ then the vertical line of integration slides to the left, picking up a residue at zero, until the integral vanishes, and if $0<x<1$ then similarly the line slides to the right, not picking up a residue. A true proof requires an estimate for integrating over truncations of the vertical line, and then estimates of the integrals over the top and bottom of rectangular contours of integration.

Note that the extraction identity holds for $\sigma>1$, the condition in the display at the end of the previous section for $s=\sigma+it$.

3. The explicit formula

Let $b>1$ be some bound, not a prime power. Conceptually, we will integrate both sides of (1) against the function b^s/s around an infinite rectangle R; the "rectangle" has right side $\{\mathrm{Re}(s)=\sigma\}$ where $\sigma>1$, and it extends infinitely high and infinitely far to the left.

Integrating the left side of (1), granting that the integrals over the top, bottom, and left sides of are zero, and then citing the identity in the previous section with $p^{-m}b$ in place of x, we get

$$\frac{1}{2\pi i}\int_R\sum_{\substack{p\\ m\geq 1}}\log p\cdot p^{-ms}\frac{b^s}{s}\,\mathrm{d}s=\sum_{\substack{p\\ m\geq 1}}\frac{\log p}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{(p^{-m}b)^s}{s}\,\mathrm{d}s=\sum_{\substack{p\\ m\geq 1\\ p^m<b}}\log p.$$

Meanwhile, the right of (1), slightly rearranged, is

$$\frac{s}{s-1}-\sum_{\rho}\frac{s}{\rho(s-\rho)}-\frac{\zeta'(0)}{\zeta(0)}+\frac{1}{2}\sum_{n\geq 1}\frac{s}{n(s+2n)}\,.$$

Its extraction integral is evaluated by summing the residues in $\{\mathrm{Re}(s)<\sigma\}$ of each term times b^s/s; e.g., for the last sum, a typical residue-term is $(1/2)b^{-2n}/n$. Equate the right side of the previous extraction calculation and the residue sum to obtain the Explicit Formula: For $b>0$ not a prime power,

$$\boxed{\sum_{\substack{p\\ m\geq 1\\ p^m<b}}\log p=b-\sum_{\rho}\frac{b^\rho}{\rho}-\frac{\zeta'(0)}{\zeta(0)}-\frac{1}{2}\log(1-b^{-2}).}$$

The terms on the right side are arranged by decreasing magnitude. Thus

$$\sum_{p^m<b}\log p\sim b,$$

and the location of the nontrivial zeros of $\zeta(s)$ determines the largest error term. Because the zeros are symmetric about the line $\mathrm{Re}(s)=1/2$, and because $|b^\rho|=b^{\mathrm{Re}(\rho)}$, the error term is as small as possible if the *Riemann hypothesis*—that $\mathrm{Re}(\rho)=1/2$ for all ρ—holds. The constant term $\zeta'(0)/\zeta(0)$ can be shown to equal $\log(2\pi)$.

4. Evaluation of $\zeta'(0)/\zeta(0)$

The logarithmic derivative of $Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is

$$\frac{Z(s)'}{Z(s)} = -\frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{\zeta'(s)}{\zeta(s)}. \tag{2}$$

As s goes to 0, the third term goes to $\zeta'(0)/\zeta(0)$, the value that we seek. As for the second term, from

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n\geq 1}(1+z/n)e^{-z/n} = ze^{\gamma z}\prod_{n\geq 1}(1+\mathcal{O}(z^2/n^2)),$$

we get

$$-\log\Gamma(z) = \log z + \gamma z + \sum_{n\geq 1}\mathcal{O}(z^2/n^2) = \log z + \gamma z + \mathcal{O}(z^2),$$

so that multiplying by -1 and then differentiating gives

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma + \mathcal{O}(z).$$

Thus the dominant terms of (2) as s goes to 0 are

$$\frac{Z(s)'}{Z(s)} \sim -\frac{1}{s} + \left(-\frac{1}{2}\log\pi - \frac{\gamma}{2} + \frac{\zeta'(0)}{\zeta(0)}\right). \tag{3}$$

Also, the logarithmic derivative of $Z(1-s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s)$ is

$$\frac{Z(1-s)'}{Z(1-s)} = \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'((1-s)/2)}{\Gamma((1-s)/2)} - \frac{\zeta'(1-s)}{\zeta(1-s)}. \tag{4}$$

For the second term, from

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n\geq 1}(1+z/n)e^{-z/n}$$

we get

$$-\log\Gamma(z) = \log z + \gamma z + \sum_{n\geq 1}(\log(1+z/n) - z/n),$$

so that differentiating gives

$$-\frac{\Gamma'(z)}{\Gamma(z)} = \frac{1}{z} + \gamma + \sum_{n\geq 1}\left(\frac{1/n}{1+z/n} - 1/n\right).$$

Divide by 2 and set $z = 1/2$ to get

$$-\frac{1}{2}\frac{\Gamma'(1/2)}{\Gamma(1/2)} = 1 + \frac{\gamma}{2} + \sum_{n\geq 1}\frac{1}{2}\left(\frac{1/n}{1+1/(2n)} - 1/n\right),$$

in which the nth summand is $-1/(2n) + 1/(2n+1)$, and so the sum is

$$-1/2 + 1/3 - 1/4 + 1/5 - \cdots = \log 2 - 1.$$

Altogether, as s goes 0 the second term of (4) goes to $\gamma/2 + \log 2$. As for the third term, from

$$\zeta(1-s) \sim \frac{1}{1-s-1} + \gamma = -\frac{1}{s} + \gamma \quad \text{as } s \to 0,$$

we get

$$-\frac{\zeta'(1-s)}{\zeta(1-s)} \sim -\frac{1}{s} - \gamma.$$

Thus the dominant terms of (4) as s goes to 0 are

$$\frac{Z(1-s)'}{Z(1-s)} \sim -\frac{1}{s} + \left(\frac{1}{2}\log\pi - \frac{\gamma}{2} + \log 2\right). \tag{5}$$

Because $Z(s)'/Z(s) = Z(1-s)'/Z(1-s)$, the constant terms on the right sides of (3) and (5) match, giving

$$\frac{\zeta'(0)}{\zeta(0)} = \log 2\pi.$$