DIRICHLET L VALUES AT NONPOSITIVE INTEGERS

(Modeled on exposition in Washington's **Cyclotomic Fields**.)

Contents

1. Basic Bernoulli numbers and polynomials

Recall the definitions of the Bernoulli numbers and the Bernoulli polynomials,

$$\frac{t}{e^t-1}=\sum_{k\geq 0}B_k\frac{t^k}{k!}\qquad\text{and}\qquad\frac{te^{Xt}}{e^t-1}=\sum_{k\geq 0}\mathbb{B}_k(X)\frac{t^k}{k!}\,.$$

Because

$$\frac{te^{Xt}}{e^t-1}=\sum_{i\geq 0}X^i\frac{t^i}{i!}\sum_{j\geq 0}B_j\frac{t^j}{j!}=\sum_{k\geq 0}\sum_{j=0}^{k}\binom{k}{j}B_jX^{k-j}\frac{t^k}{k!}$$

the Bernoulli polynomials are

$$\mathbb{B}_k(X)=\sum_{j=0}^{k}\binom{k}{j}B_jX^{k-j},\quad k\geq 0.$$

Arguably it would be better to take $t/(1-e^{-t})=te^t/(e^t-1)=t/(e^t-1)+t$ instead as the definition of the Bernoulli number generating function $\sum_k B_kt^k/k!$, the only effect being to modify B_1 from $-1/2$ to $1/2$, but the stated definition is entrenched. Opting between the definitions is a matter of deciding whether one deems it more natural to count from 1 to n or from 0 to $n-1$.

Because

$$\frac{t}{e^t-1}+t=\frac{te^t}{e^t-1}=\frac{-t}{e^{-t}-1}$$

we have for all $k\geq 0$,

$$\mathbb{B}_k(0)+\delta_{k,1}=\mathbb{B}_k(1)=(-1)^k\mathbb{B}_k(0).$$

The relation $\mathbb{B}_k(X)=\sum_{j=0}^{k}\binom{k}{j}B_jX^{k-j}$ specializes to $\mathbb{B}_k(1)=\sum_{j=0}^{k}\binom{k}{j}B_j$. The fact that this equals $\mathbb{B}_k(0)=B_k$ except when $k=1$ is the defining condition of the Bernoulli numbers, $t=(e^t-1)\sum_{k\geq 0}B_kt^k/k!$; indeed, this condition is

$$t=\sum_{i\geq 1}\frac{t^i}{i!}\sum_{j\geq 0}B_j\frac{t^j}{j!}=\sum_{k\geq 1}\sum_{j=0}^{k-1}\binom{k}{j}B_k\frac{t^k}{k!}\,.$$

That is, $B_0 = 1$ and then $\sum_{j=0}^{k-1} \binom{k}{j} B_k = 0$ for $k \geq 2$. This lets us compute the Bernoulli numbers handily.

The Bernoulli polynomials have a sort of averaging property, as follows. For any positive integer m, the Bernoulli polynomial definition and the finite geometric sum formula give

$$\sum_{k\geq 0} \mathbb{B}_k(X)\frac{t^k}{k!} = \frac{te^{Xt}}{e^t - 1} \qquad \text{and} \qquad \frac{1}{e^t-1} = \frac{1}{e^{mt}-1}\sum_{j=0}^{m-1} e^{jt}$$

and consequently

$$\sum_{k\geq 0} \mathbb{B}_k(X)\frac{t^k}{k!} = \frac{1}{m}\sum_{j=0}^{m-1} \frac{mte^{(X+j)/m\cdot mt}}{e^{mt}-1} = \sum_{k\geq 0}\sum_{j=0}^{m-1} m^{k-1}\mathbb{B}_k(\tfrac{X+j}{m})\frac{t^k}{k!}$$

which is to say,

$$\mathbb{B}_k(X) = m^{k-1}\sum_{j=0}^{m-1} \mathbb{B}_k(\tfrac{X+j}{m}), \quad k = 0, 1, 2, \dots . \tag{1}$$

We will use this relation below.

2. Dirichlet character Bernoulli numbers

Let χ be a Dirichlet character of conductor N. The generating function definitions of the χ-Bernoulli numbers $B_{k,\chi}$ and the Bernoulli polynomials $\mathbb{B}_k(X)$ are

$$\sum_{k\geq 0} B_{k,\chi}\frac{t^k}{k!} = \sum_{a=0}^{N-1} \chi(a)\frac{te^{at}}{e^{Nt}-1} \qquad \text{and} \qquad \frac{te^{Xt}}{e^t-1} = \sum_{k\geq 0} \mathbb{B}_k(X)\frac{t^k}{k!}$$

and it follows that

$$\sum_{k\geq 0} B_{k,\chi}\frac{t^k}{k!} = \frac{1}{N}\sum_{a=0}^{N-1} \chi(a)\frac{Nte^{a/N\cdot Nt}}{e^{Nt}-1} = \sum_{k\geq 0} N^{k-1}\sum_{a=0}^{N-1}\chi(a)\mathbb{B}_k(\tfrac{a}{N})\frac{t^k}{k!}$$

so that each χ-Bernoulli number is a weighted average of Bernoulli polynomial values,

$$B_{k,\chi} = N^{k-1}\sum_{a=0}^{N-1} \chi(a)\mathbb{B}_k(\tfrac{a}{N}), \quad k = 0, 1, 2, \dots . \tag{2}$$

Now let $M = QN$ be an integer multiple of the conductor. We show that if N is replaced by its multiple M in the right side of the previous display then the result is still $B_{k,\chi}$. Each a from 0 to $M-1$ is uniquely $a = qN + a'$ with $0 \leq q < Q$ and

$0 \le a' < N$. Compute for any nonnegative integer k,

$$
\begin{aligned}
M^{k-1}\sum_{a=0}^{M-1}\chi(a)\mathbb{B}_k(\tfrac{a}{M}) &= (QN)^{k-1}\sum_{q=0}^{Q-1}\sum_{a'=0}^{N-1}\chi(qN+a')\mathbb{B}_k(\tfrac{qN+a'}{QN})\\
&= N^{k-1}\sum_{a'=0}^{N-1}\chi(a')Q^{k-1}\sum_{q=0}^{Q-1}\mathbb{B}_k(\tfrac{a'/N+q}{Q})\\
&= N^{k-1}\sum_{a'=0}^{N-1}\chi(a')\mathbb{B}_k(\tfrac{a'}{N}) \quad \text{by (1)}\\
&= B_{k,\chi} \quad \text{by (2)}.
\end{aligned}
$$

Returning to the χ-Bernoulli number definition

$$\sum_{k\ge 0} B_{k,\chi}\frac{t^k}{k!} = \sum_{a=0}^{N-1}\chi(a)\frac{te^{at}}{e^{Nt}-1},$$

note that when χ is trivial, so that $N = 1$, this is not the same thing as summing over a from 1 to N: the previous display has $t/(e^t - 1)$ on the right side, whereas the other way would be $te^t/(e^t - 1)$. These are exactly the two definitions of the basic Bernoulli numbers, which is to say that by our definitions $B_1 = -1/2$ but $B_{1,1} = 1/2$.

Assuming that χ is nontrivial, so that $N > 1$ and $\chi(0) = 0$, replace t by $-t$ in the right side of the previous display to get

$$-\sum_{a=1}^{N-1}\chi(a)\frac{te^{-at}}{e^{-Nt}-1} = \operatorname{sgn}(\chi)\sum_{a=1}^{N-1}\chi(N-a)\frac{te^{(N-a)t}}{e^{Nt}-1} = \operatorname{sgn}(\chi)\sum_{a=1}^{N}\chi(a)\frac{te^{at}}{e^{Nt}-1}.$$

This shows that if χ is even then all $B_{k,\chi}$ for odd k are zero, and if χ is odd then all $B_{k,\chi}$ for even k are zero.

The χ-Bernoulli numbers can be computed iteratively in the same fashion as the basic Bernoulli numbers. Indeed, the relation

$$\sum_{k\ge 0} B_{k,\chi}\frac{t^k}{k!} = \sum_{a=0}^{N-1}\chi(a)\frac{te^{at}}{e^{Nt}-1}$$

is, multiplying through by the denominator of the right side,

$$\sum_{j\ge 1} N^j\frac{t^j}{j!}\sum_{k\ge 0} B_{k,\chi}\frac{t^k}{k!} = \sum_{a=0}^{N-1}\chi(a)\sum_{n\ge 0} a^n\frac{t^{n+1}}{n!},$$

or

$$\sum_{n\ge 1}\sum_{k=0}^{n-1}\binom{n}{k}N^{n-k}B_{k,\chi}\frac{t^n}{n!} = \sum_{n\ge 1} n\sum_{a=0}^{N-1}\chi(a)a^{n-1}\frac{t^n}{n!},$$

so that

$$\sum_{k=0}^{n-1}\binom{n}{k}N^{n-k}B_{k,\chi} = n\sum_{a=0}^{N-1}\chi(a)a^{n-1}, \quad n = 1, 2, 3, \dots .$$

(If $\chi = 1$ then this is $\sum_{k=0}^{n-1}\binom{n}{k}B_k = n \cdot 0^{n-1}$ and the right side is 1 for $n = 1$ and otherwise 0.) Assuming that χ is nontrivial, the previous display with $n = 1$ gives $NB_{0,\chi} = \sum_a \chi(a)a^0 = 0$ so that

$$B_{0,\chi} = 0 \quad (\chi \text{ nontrivial}),$$

and then $n = 2$ gives $N^2 B_{0,\chi} + 2NB_{1,\chi} = 2\sum_a \chi(a)a$ so that

$$B_{1,\chi} = \frac{1}{N}\sum_{a=0}^{N-1}\chi(a)a \quad (\chi \text{ nontrivial}).$$

We will use this formula at the end of this writeup.

3. Hurwitz zeta function and its continuation

For any positive real number r,

$$\Gamma(s)r^{-s} = \int_{t=0}^{\infty} e^{-rt}t^s \frac{\mathrm{d}t}{t}, \quad \mathrm{Re}(s) > 1,$$

and so, with the Hurwitz zeta function, defined as

$$\zeta(s,b) = \sum_{n\geq 0}(n+b)^{-s}, \quad \mathrm{Re}(s) > 1,\ 0 < b \leq 1,$$

we have

$$\begin{aligned}\Gamma(s)\zeta(s,b) &= \sum_{n\geq 0}\int_{t=0}^{\infty} e^{-(n+b)t}t^s\frac{\mathrm{d}t}{t} = \int_{t=0}^{\infty}\sum_{n\geq 0} e^{-nt}e^{-bt}t^s\frac{\mathrm{d}t}{t} \\ &= \int_{t=0}^{\infty}\frac{e^{-bt}}{1-e^{-t}}t^s\frac{\mathrm{d}t}{t} = \int_{t=0}^{\infty}\frac{te^{(1-b)t}}{e^t-1}t^{s-2}\,\mathrm{d}t.\end{aligned}$$

It follows that for $0 < \varepsilon < 1$ and H_ε the Hankel contour,

$$\zeta(s,b) = \frac{1}{\Gamma(s)(e^{2\pi i s}-1)}\int_{H_\varepsilon}\frac{ze^{(1-b)z}}{e^z-1}z^{s-2}\,\mathrm{d}z, \quad \mathrm{Re}(s) > 1.$$

The equality just given extends $\zeta(s,b)$ meromorphically to $\mathbb{C}$.

At $s = 1-k$ for $k = 1,2,3,\dots$ we have $\Gamma(s)(e^{2\pi i s}-1) = 2\pi i(-1)^{k-1}/(k-1)!$, and so

$$\begin{aligned}\zeta(1-k,b) &= (-1)^{k-1}(k-1)!\frac{1}{2\pi i}\int_{H_\varepsilon}\frac{ze^{(1-b)z}}{e^z-1}z^{-k-1}\,\mathrm{d}z \\ &= (-1)^{k-1}(k-1)!\,\mathrm{Res}_{z=0}\left(\frac{ze^{(1-b)z}}{e^z-1}z^{-k-1}\right), \quad k = 1,2,3,\dots.\end{aligned}$$

And because

$$\frac{ze^{(1-b)z}}{e^z-1}z^{-k-1} = \sum_{\ell\geq 0}\mathbb{B}_\ell(1-b)\frac{z^{\ell-k-1}}{\ell!},$$

the residue is $\mathbb{B}_k(1-b)/k!$. Thus

$$\zeta(1-k,b) = \frac{(-1)^{k-1}}{k}\mathbb{B}_k(1-b), \quad k = 1,2,3,\dots.$$

4. Dirichlet L at nonpositive integers

Let χ be a Dirichlet character. We evaluate $L(\chi, 1-k)$ for $k = 1, 2, 3, \dots$. The Dirichlet L-function is a weighted average of Hurwitz zeta function values,

$$L(\chi, s) = \sum_{a=1}^{N} \chi(a) N^{-s} \zeta(s, a/N),$$

and this determines $L(\chi, 1-k)$ to be essentially $B_{k,\chi}$,

$$\begin{aligned} L(\chi, 1-k) &= \frac{(-1)^{k-1}}{k} \sum_{a=1}^{N} \chi(a) N^{k-1} \mathbb{B}_k(1 - a/N) \\ &= \frac{(-1)^{k-1}\mathrm{sgn}(\chi)}{k} \sum_{a=0}^{N-1} \chi(a) N^{k-1} \mathbb{B}_k(a/N) \quad \text{(replacing } a \text{ with } N-a) \\ &= \frac{(-1)^{k-1}\mathrm{sgn}(\chi) B_{k,\chi}}{k}, \quad k = 1, 2, 3, \dots. \end{aligned}$$

If χ is even then $B_{k,\chi} = 0$ for odd k (except for the special case $(\chi, k) = (1, 1)$) and so $(-1)^{k-1}\mathrm{sgn}(\chi) B_{k,\chi} = -B_{k,\chi}$, and if χ is odd then $B_{k,\chi} = 0$ for even k and again $(-1)^{k-1}\mathrm{sgn}(\chi) B_{k,\chi} = -B_{k,\chi}$. So finally,

$$\boxed{L(\chi, 1-k) = -\frac{B_{k,\chi}}{k}, \quad k = 1, 2, 3, \dots \qquad \text{(excluding } (\chi, k) = (1, 1)).}$$

In the special case $(\chi, k) = (1, 1)$,

$$\zeta(0) = -\tfrac{1}{2}.$$

The boxed equality subsumes this if we take $B_1 = 1/2$.

5. Odd quadratic case

Let $\chi = \overline{\chi}$ be an odd quadratic character of conductor N. We have learned that its Gauss sum $\tau(\chi)$ is $iN^{1/2}$. Suppose that $s \sim 0$. Then $\Gamma(s) \sim 1/s$ and $\cos(\pi(s-1)/2) \sim \pi s/2$, and so from our writeup on continuations and functional equations,

$$L(\chi, 1-s) = \frac{2i}{\tau(\chi)} \left(\frac{2\pi}{N}\right)^{-s} \Gamma(s) \cos\left(\frac{\pi(s-1)}{2}\right) L(\chi, s) \sim \frac{2}{N^{1/2}} \frac{1}{s} \frac{\pi s}{2} L(\chi, s).$$

This and then the previous boxed formula with $k = 1$ give

$$L(\chi, 1) = \frac{\pi}{N^{1/2}} L(\chi, 0) = -\frac{\pi}{N^{1/2}} B_{1,\chi},$$

which is to say, by the computation of $B_{1,\chi}$ earlier in this writeup,

$$\boxed{L(\chi, 1) = -\frac{\pi}{N^{3/2}} \sum_{a=0}^{N-1} \chi(a) a, \quad \chi \text{ odd quadratic.}}$$

Later in the semester we will see that this L-value plays an important role in the theory of imaginary quadratic fields.