FOURIER TRANSFORM OF THE GAUSSIAN

The (one-dimensional) Gaussian function is

$$g : \mathbb{R} \longrightarrow \mathbb{R}, \quad g(x) = e^{-x^2/2},$$

and it is characterized by the conditions

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} g(x) = -x\, g(x), \quad g(0) = 1.}$$

For any $\xi \in \mathbb{R}$ the frequency-ξ oscillation is

$$\psi_\xi : \mathbb{R} \longrightarrow \mathbb{C}, \quad \psi_\xi(x) = e^{i\xi x}.$$

The Fourier transform of the Gaussian is, with $\mathrm{d}\mu(x) = (2\pi)^{-1/2}\, \mathrm{d}x$,

$$\mathcal{F}g : \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathcal{F}g(\xi) = \int_{\mathbb{R}} g(x)\overline{\psi}_\xi(x)\, \mathrm{d}\mu(x).$$

Note that $\mathcal{F}g$ is real-valued because g is even. We have the derivatives

$$\begin{aligned}
\frac{\partial}{\partial \xi} \overline{\psi}_\xi(x) &= -ix\, \overline{\psi}_\xi(x), \\
\frac{\mathrm{d}}{\mathrm{d}x} g(x) &= -x\, g(x), \\
\frac{\partial}{\partial x} \overline{\psi}_\xi(x) &= -i\xi\, \overline{\psi}_\xi(x).
\end{aligned}$$

To study the Fourier transform of the Gaussian, differentiate under the integral sign, then use the first two equalities in the previous display, then integrate by parts, then use the third equality in the previous display,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi}(\mathcal{F}g)(\xi) &= \int_{\mathbb{R}} g(x) \frac{\partial}{\partial \xi} \overline{\psi}_\xi(x)\, \mathrm{d}\mu(x) \\
&= i \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}x} g(x) \overline{\psi}_\xi(x)\, \mathrm{d}\mu(x) \\
&= -i \int_{\mathbb{R}} g(x) \frac{\partial}{\partial x} \overline{\psi}_\xi(x)\, \mathrm{d}\mu(x) \\
&= -\xi \int_{\mathbb{R}} g(x)\overline{\psi}_\xi(x)\, \mathrm{d}\mu(x) \\
&= -\xi\, (\mathcal{F}g)(\xi).
\end{aligned}$$

Also, $\mathcal{F}g(0) = 1$. Thus $\mathcal{F}g$ satisfies the characterizing properties of the Gaussian. That is, the Gaussian is its own Fourier transform,

$$\mathcal{F}g = g.$$