Why does the Fourier series of a continuous function mean-square converge to the function?

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The issue

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be $\mathbb{Z}$-periodic and continuous

The Fourier coefficients of f are

$$\widehat{f}(n) = \int_{t=0}^{1} f(t)e^{-2\pi int}\,\mathrm{d}t, \quad n \in \mathbb{Z}$$

The partial sums of its Fourier series are

$$(s_N f)(x) = \sum_{|n|\leq N} \widehat{f}(n)e^{2\pi inx}, \quad N \in \mathbb{Z}_{\geq 0}$$

The partial sums $s_N f$ *mean-square converge* to f

$$\lim_N \int_{x=0}^{1} |(s_N f)(x) - f(x)|^2\,\mathrm{d}x = 0$$

How is this true, even though no claim about pointwise convergence of the $s_N f$ to f is supportable?

Outline

1. One physical space, two complete function spaces
2. Trigonometric polynomials
3. The zoom-in construction and trigonometric approximation
4. Fourier polynomials
5. Féjer polynomials
6. Féjer polynomial approximation is good in $\mathcal{C}^0$
7. Fourier polynomial approximation is good in $\mathcal{L}^2$
8. A pointwise convergence result

Outline

The physical space and the test function space

The physical space is $X = \mathbb{R}/\mathbb{Z}$

- compact topological group
- inherits its topology and Haar measure from $\mathbb{R}$

The test function space is $\mathcal{C}^\infty(X)$ (smooth functions from X to $\mathbb{C}$)

Two norms on the test function space

$$\text{uniform norm} \quad |f|_{\mathcal{C}^0} = \sup_{x \in X} |f(x)|$$

$$\text{mean-square norm} \quad |f|_{\mathcal{L}^2} = \left(\int_X |f(x)|^2 \right)^{1/2}$$

The uniform norm dominates the mean-square norm

$$|f|_{\mathcal{C}^0} \geq |f|_{\mathcal{L}^2}$$

So the identity set-map

$$\left(\mathcal{C}^\infty(X), |\cdot|_{\mathcal{C}^0}\right) \longrightarrow \left(\mathcal{C}^\infty(X), |\cdot|_{\mathcal{L}^2}\right)$$

is continuous

Two completions of the test function space

$$\mathcal{C}^0(X) = \text{completion of } \big(\mathcal{C}^\infty(X), |\cdot|_{\mathcal{C}^0}\big)$$
$$\mathcal{L}^2(X) = \text{completion of } \big(\mathcal{C}^\infty(X), |\cdot|_{\mathcal{L}^2}\big)$$

So a continuous map

$$\mathcal{C}^0(X) \longrightarrow \mathcal{L}^2(X)$$

of *complete* spaces amenable to analysis

$\mathcal{L}^2(X)$ carries an inner product

$$\langle f, g\rangle = \int_X f\overline{g}$$

The setting

We consider functions in $\mathcal{C}^0(X)$ from now on

freely viewing them as elements of $\mathcal{L}^2(X)$ as well

Outline

Orthonormal family of characters in $\mathcal{L}^2$

The *integer-frequency oscillations* are

$$\{\psi_n : n \in \mathbb{Z}\}$$

where

$$\psi_n : X \longrightarrow \mathbb{C}^\times \qquad \text{is} \qquad \psi_n(x) = e^{2\pi inx}$$

Characters

$$\psi_n(x+y) = \psi_n(x)\psi_n(y)$$

Orthonormal

$$\langle \psi_n, \psi_m \rangle = \delta_{n,m} \quad \text{(Kronecker delta)}$$

Handy properties

$$\psi_n = \psi_1^n \qquad \psi_n\psi_m = \psi_{n+m} \qquad \psi_n(-x) = \overline{\psi}_n(x) = \psi_{-n}(x)$$

Fourier polynomials

In $\mathcal{L}^2(X)$ the Nth Fourier polynomial of any f is

$$s_N f = \sum_{|n| \leq N} \langle f, \psi_n \rangle \psi_n$$

As earlier, the Fourier coefficients are

$$\langle f, \psi_n \rangle = \int_{t=0}^{1} f(t) e^{-2\pi i n t} \, \mathrm{d}t$$

Especially the value at 0 is just the sum of the coefficients, a single inner product

$$(s_N f)(0) = \sum_{|n| \leq N} \langle f, \psi_n \rangle = \langle f, \sum_{|n| \leq N} \psi_n \rangle$$

The Fourier polynomials are $\mathcal{L}^2$-best

Fix $f \in \mathcal{C}^0(X)$, fix N

$s_N f$ is constructed to make $f - s_N f$ orthogonal to the span of $\{\psi_n : |n| \leq N\}$

For any t_N in this span, also $s_N f - t_N$ is in the span, and the orthogonal decomposition

$$f - t_N = (f - s_N f) + (s_N f - t_N)$$

gives

$$|f - t_N|^2_{\mathcal{L}^2} = |f - s_N f|^2_{\mathcal{L}^2} + |s_N f - t_N|^2_{\mathcal{L}^2}$$

So

$$|f - t_N|^2_{\mathcal{L}^2} \geq |f - s_N f|^2_{\mathcal{L}^2}$$

$s_N f$ is the $\mathcal{L}^2$-best degree-N trigonometric polynomial approximation of f
But is "$\mathcal{L}^2$-best" any good at all?

General trigonometric polynomials

Let $a = \{a_n\}_{n\in\mathbb{Z}}$ be compactly supported, with $a_0 = 1$

The corresponding trigonometric polynomial-formation operator $t = t_a$ is

$$tf = \sum_n a_n \langle f, \psi_n \rangle \psi_n$$

a_n: the weight given the nth term of the Fourier series

And

$$(tf)(0) = \sum_n a_n \langle f, \psi_n \rangle = \langle f, \sum_n a_n \psi_n \rangle$$

Outline

Zoom-in construction

Given $f \in \mathcal{C}^0(X)$ and $y \in X$

The zoom-in of f at y is just f in local coordinates about y

$$f_y(x) = f(y + x) - f(y)$$

Note $f_y(0) = 0$

Functional notation

$$f_y = T_y f - f(y) \qquad T_y = \text{pre-translate by } y$$

Trig-polynomial approximation preserved under zoom-in

Compute for any y

$$(tf)(y) = \sum_n a_n \langle f, \psi_n \rangle \psi_n(y) = \sum_n a_n \langle f, T_{-y}\psi_n \rangle = \sum_n a_n \langle T_y f, \psi_n \rangle$$

That is

$$(tf)(y) = (tT_y f)(0)$$

Also, viewing $f(y)$ as a constant function, $t(f(y)) = f(y)$ is the same constant function
So again letting $f_y = T_y f - f(y)$ denote the zoom-in of f at y,

$$(tf)(y) - f(y) = (tT_y f)(0) - t(f(y))(0) = t(T_y f - f(y))(0) = (tf_y)(0)$$

So, to check trigonometric polynomial approximation anywhere, just work with the zoom-in at 0

Outline

Fourier polynomial in particular

$$s_N f = \sum_{|n|\le N} \langle f, \psi_n\rangle \psi_n$$

Fix y, let f_y be the zoom-in

$$(s_N f - f)(y) = (s_N f_y)(0) = \langle f_y, \sum_{|n|\le N} \psi_n\rangle$$

So need to study

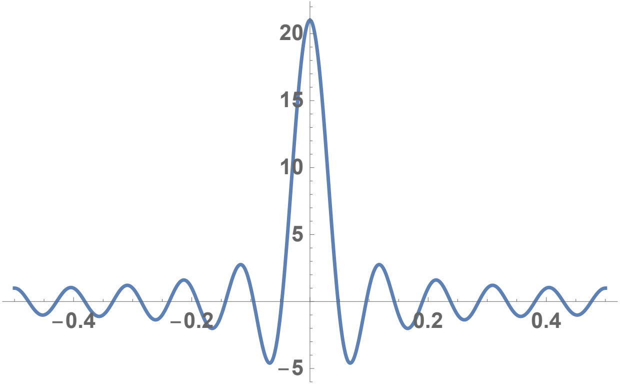
$$D_N = \sum_{|n|\le N} \psi_n$$

This is the *Dirichlet kernel*

Dirichlet kernel

$$D_N = \sum_{|n| \le N} \psi_n = \frac{\psi_{N+1} - \psi_{-N}}{\psi_1 - 1} = \frac{\psi_{N+1/2} - \psi_{-N-1/2}}{\psi_{1/2} - \psi_{-1/2}} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$$

E.g., for $N = 10$



Dirichlet kernel properties

The Dirichlet kernel is

- good in $\mathcal{L}^2$, *but this is hard for us to see because we don't have visual* $\mathcal{L}^2$ *intuition*
- *bad in* $\mathcal{C}^0$, because of positive and negative values, horizontal spread

Outline

$\mathcal{C}^0$-improving the kernel: Dirichlet to Féjer

Guided by faith that the Dirichlet kernel is $\mathcal{L}^2$-good, i.e., mean-square good, repair its $\mathcal{C}^0$-deficiencies by *squaring* it

Example

$$D_1^2 = (\psi_{-1} + \psi_0 + \psi_1)(\psi_{-1} + \psi_0 + \psi_1) = \psi_{-2} + 2\psi_{-1} + 3\psi_0 + 2\psi_1 + \psi_2$$

And in general

$$D_N^2 = \sum_{|n| \le 2N} (2N + 1 - |n|)\psi_n$$

Low-frequency coefficients are larger

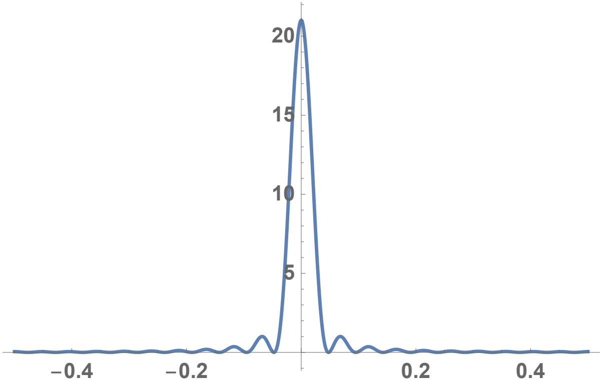
$$1, \quad 2, \quad 3, \quad \ldots, \quad 2N, \quad 2N+1, \quad 2N, \quad \ldots, \quad 3, \quad 2, \quad 1$$

Féjer kernel

Scale the constant coefficient to 1

$$K_N = \frac{1}{2N+1} D_N^2 = \frac{1}{2N+1} \frac{\sin^2((2N+1)\pi x)}{\sin^2(\pi x)}$$

E.g., for $N = 10$



Good $\mathcal{C}^0$-properties of the Féjer kernel

The Féjer kernel is

- positive: $K_N \geq 0$ (because K_N is a scaled square)
- normalized: $\int_X K_N = 1$ (because all ψ_n integrate to 0 except ψ_0)
- concentrated near 0: For all $\varepsilon, \delta > 0$, there exists N_o such that

$$\int_{|x| \geq \delta} K_N < \varepsilon \quad \text{for all } N \geq N_o$$

(modeling X as $[-1/2, 1/2]$)

Establishing the third bullet is the key, and a good exercise

Féjer polynomials

The Nth Féjer polynomial is

$$t_N f = \sum_{|n| \le 2N} \frac{2N+1-|n|}{2N+1} \langle f, \psi_n \rangle \psi_n$$

For fixed freqency n, the nth Féjer polynomial coefficient isn't static as the degree N grows, but it does converge to the Fourier coefficient

$$\frac{2N+1-|n|}{2N+1} \langle f, \psi_n \rangle \xrightarrow{N} \langle f, \psi_n \rangle$$

The Nth Féjer polynomial at 0 of the zoom-in f_y for any $y \in X$ is just the inner product against the Féjer kernel

$$(t_N f_y)(0) = \langle f_y, K_N \rangle$$

Outline

Uniform Féjer polynomial approximation is good

Let $f \in \mathcal{C}^0(X)$ be given

Let $\varepsilon > 0$ be given

Set $\varepsilon' = \varepsilon/(1 + 2|f|_{\mathcal{C}^0})$

Note f is uniformly continuous because X is compact
so there exists suitable $\delta = \delta(\varepsilon', f)$

For any $y \in X$, let $g = f_y$ be the zoom-in

- δ from above works for g at 0, independently of y
- $|g|_{\mathcal{C}^0} \leq 2|f|_{\mathcal{C}^0}$, independently of y

(continued on next slide)

Uniform Féjer polynomial approximation (continued)

For $N \geq N_o(f)$, $(t_N f - f)(y)$ is $(t_N g)(0)$, which is $\langle g, K_N \rangle$,

$$
\begin{aligned}
|(t_N f - f)(y)| &= \left| \int_X g K_N \right| \\
&\leq \int_{|x| \leq \delta} |g| K_N + \int_{|x| \geq \delta} |g| K_N \quad (K_N \text{ positive}) \\
&< \varepsilon' + |g|_{\mathcal{C}^0} \varepsilon' \qquad (K_N \text{ normalized, concentrated}) \\
&\leq (1 + 2|f|_{\mathcal{C}^0}) \varepsilon' \\
&= \varepsilon, \ \text{independently of } y
\end{aligned}
$$

Given $\varepsilon > 0$, there exists $N_o(f)$ such that $|f - t_N f|_{\mathcal{C}^0} < \epsilon$ for all $N \geq N_o(f)$

We want close approximation of f by Fourier polynomials in $\mathcal{L}^2$

We have close approximation of f by the wrong polynomials in the wrong space

Outline

Mean-square Fourier polynomial approximation is good

Given f in $\mathcal{C}^0(X)$

Given $\varepsilon > 0$

Recall

- The Fourier polynomial $s_{2N}f$ is the $\mathcal{L}^2$-best degree-$2N$ trigonometric polynomial approximation of f, by general inner product space principles
- The uniform $\mathcal{C}^0$-norm dominates the mean-square $\mathcal{L}^2$-norm

So for $N \geq N_o(f)$,

$$|f - s_{2N}f|_{\mathcal{L}^2} \leq |f - t_Nf|_{\mathcal{L}^2} \leq |f - t_Nf|_{\mathcal{C}^0} < \varepsilon$$

That is, no claim is made that $\{(s_Nf)(y)\} \xrightarrow{N} f(y)$ for any particular y, but rather

$$\boxed{\{s_Nf\} \xrightarrow{N} f \text{ in } \mathcal{L}^2(X) \quad \text{because} \quad \{t_Nf\} \xrightarrow{N} f \text{ in } \mathcal{C}^0(X)}$$

The Fourier polynomials mean-square converge to f because the Féjer polynomials uniformly converge to f

Outline

Zoom in to study pointwise convergence too

Fourier polynomial approximation is preserved under zoom-in

$$(s_N f - f)(y) = (s_N f_y)(0) = \langle f_y, D_N \rangle$$

That is, with $g = f_y$ as usual, we need to study

$$\int_{x=-1/2}^{1/2} \frac{x}{e^{2\pi i x} - 1} \cdot \frac{g(x)}{x} \cdot \left(e^{2\pi i (N+1)x} - e^{-2\pi i N x} \right) \mathrm{d}x$$

The first quotient in the integrand is well-behaved at 0

If so is the second, then the Riemann-Lebesgue lemma says that the integral goes to 0 as N grows. So

> *If f has left and right derivatives at y, not necessarily equal, then $\{(s_N f)(y)\} \xrightarrow{N} f(y)$*