Why does the Fourier series of a continuous function mean-square converge to the function?

Jerry Shurman

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The issue

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How is this true, even though no claim about pointwise convergence of the $s_N f$ to f is supportable?

Outline

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The test function space is $\mathcal{C}^\infty(X)$ (smooth functions from X to $\mathbb{C}$)

Two norms on the test function space

$$\text{uniform norm} \quad |f|_{\mathcal{C}^0} = \sup_{x \in X} |f(x)|$$

$$\text{mean-square norm} \quad |f|_{\mathcal{L}^2} = \left(\int_X |f(x)|^2 \right)^{1/2}$$

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So the identity set-map

$$(\mathcal{C}^\infty(X), |\cdot|_{\mathcal{C}^0}) \longrightarrow (\mathcal{C}^\infty(X), |\cdot|_{\mathcal{L}^2})$$

is continuous

Two completions of the test function space

$$\mathcal{C}^0(X) = \text{completion of } \big(\mathcal{C}^\infty(X), |\cdot|_{\mathcal{C}^0}\big)$$
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$\mathcal{L}^2(X)$ carries an inner product

$$\langle f, g\rangle = \int_X f\overline{g}$$

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freely viewing them as elements of $\mathcal{L}^2(X)$ as well

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Orthonormal family of characters in $\mathcal{L}^2$

The *integer-frequency oscillations* are

$$\{\psi_n : n \in \mathbb{Z}\}$$

where

$$\psi_n : X \longrightarrow \mathbb{C}^\times \qquad \text{is} \qquad \psi_n(x) = e^{2\pi i n x}$$

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Handy properties

$$\psi_n = \psi_1^n \qquad \psi_n\psi_m = \psi_{n+m} \qquad \psi_n(-x) = \overline{\psi}_n(x) = \psi_{-n}(x)$$

Fourier polynomials

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Especially the value at 0 is just the sum of the coefficients, a single inner product

$$(s_N f)(0) = \sum_{|n| \leq N} \langle f, \psi_n \rangle = \langle f, \sum_{|n| \leq N} \psi_n \rangle$$

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For any t_N in this span, also $s_N f - t_N$ is in the span, and the orthogonal decomposition

$$f - t_N = (f - s_N f) + (s_N f - t_N)$$

gives

$$|f - t_N|^2_{\mathcal{L}^2} = |f - s_N f|^2_{\mathcal{L}^2} + |s_N f - t_N|^2_{\mathcal{L}^2}$$

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But is "$\mathcal{L}^2$-best" any good at all?

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And

$$(tf)(0) = \sum_n a_n \langle f, \psi_n \rangle = \langle f, \sum_n a_n \psi_n \rangle$$

Outline

Zoom-in construction

Given $f \in \mathcal{C}^0(X)$ and $y \in X$

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Note $f_y(0) = 0$

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Functional notation

$$f_y = T_y f - f(y) \qquad T_y = \text{pre-translate by } y$$

Trig-polynomial approximation preserved under zoom-in

Compute for any y

$$(tf)(y) = \sum_n a_n \langle f, \psi_n \rangle \psi_n(y) = \sum_n a_n \langle f, T_{-y}\psi_n \rangle = \sum_n a_n \langle T_y f, \psi_n \rangle$$

That is

$$(tf)(y) = (tT_y f)(0)$$

Also, viewing $f(y)$ as a constant function, $t(f(y)) = f(y)$ is the same constant function

So again letting $f_y = T_y f - f(y)$ denote the zoom-in of f at y,

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So, to check trigonometric polynomial approximation anywhere, just work with the zoom-in at 0

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So need to study

$$D_N = \sum_{|n| \leq N} \psi_n$$

This is the *Dirichlet kernel*

Dirichlet kernel

$$D_N = \sum_{|n|\le N} \psi_n = \frac{\psi_{N+1} - \psi_{-N}}{\psi_1 - 1} = \frac{\psi_{N+1/2} - \psi_{-N-1/2}}{\psi_{1/2} - \psi_{-1/2}} = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}$$

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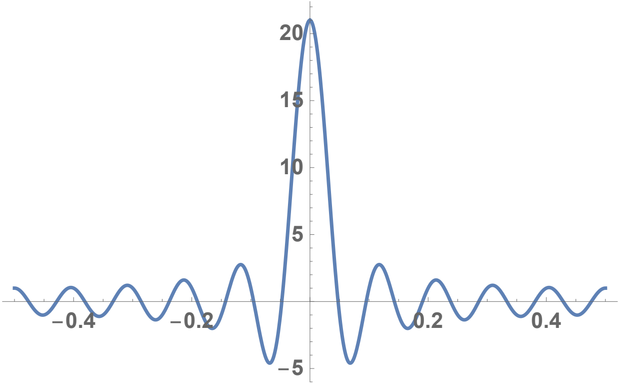
E.g., for $N = 10$



Dirichlet kernel properties

The Dirichlet kernel is

- good in $\mathcal{L}^2$, *but this is hard for us to see because we don't have visual* $\mathcal{L}^2$ *intuition*
- *bad in* $\mathcal{C}^0$, because of positive and negative values, horizontal spread

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Example

$$D_1^2 = (\psi_{-1} + \psi_0 + \psi_1)(\psi_{-1} + \psi_0 + \psi_1) = \psi_{-2} + 2\psi_{-1} + 3\psi_0 + 2\psi_1 + \psi_2$$

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And in general

$$D_N^2 = \sum_{|n| \leq 2N} (2N + 1 - |n|)\psi_n$$

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Low-frequency coefficients are larger

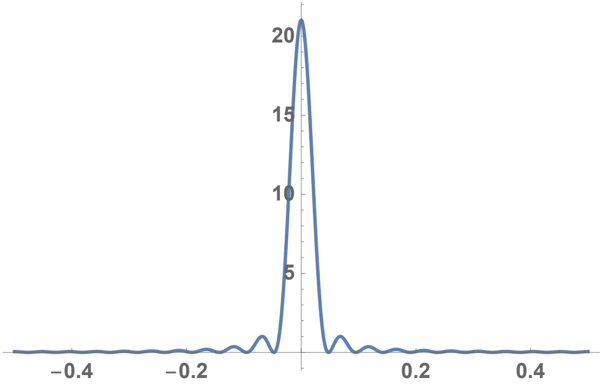
$$1, \quad 2, \quad 3, \quad \ldots, \quad 2N, \quad 2N+1, \quad 2N, \quad \ldots, \quad 3, \quad 2, \quad 1$$

Féjer kernel

Scale the constant coefficient to 1

$$K_N = \frac{1}{2N+1} D_N^2 = \frac{1}{2N+1} \frac{\sin^2((2N+1)\pi x)}{\sin^2(\pi x)}$$

E.g., for $N = 10$



Good $\mathcal{C}^0$-properties of the Féjer kernel

The Féjer kernel is

- positive: $K_N \geq 0$ (because K_N is a scaled square)
- normalized: $\int_X K_N = 1$ (because all ψ_n integrate to 0 except ψ_0)
- concentrated near 0: For all $\varepsilon, \delta > 0$, there exists N_o such that

$$\int_{|x| \geq \delta} K_N < \varepsilon \quad \text{for all } N \geq N_o$$

(modeling X as $[-1/2, 1/2]$)

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Establishing the third bullet is the key, and a good exercise

Féjer polynomials

The Nth Féjer polynomial is

$$t_N f = \sum_{|n| \leq 2N} \frac{2N + 1 - |n|}{2N + 1} \langle f, \psi_n \rangle \psi_n$$

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For fixed freqency n, the nth Féjer polynomial coefficient isn't static as the degree N grows, but it does converge to the Fourier coefficient

$$\frac{2N+1-|n|}{2N+1} \langle f, \psi_n \rangle \xrightarrow{N} \langle f, \psi_n \rangle$$

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The Nth Féjer polynomial at 0 of the zoom-in f_y for any $y \in X$ is just the inner product against the Féjer kernel

$$(t_N f_y)(0) = \langle f_y, K_N \rangle$$

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- $|g|_{\mathcal{C}^0} \leq 2|f|_{\mathcal{C}^0}$, independently of y

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(continued on next slide)

Uniform Féjer polynomial approximation (continued)

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$$
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|(t_N f - f)(y)| &= \left| \int_X g K_N \right| \\
&\leq \int_{|x| \leq \delta} |g| K_N + \int_{|x| \geq \delta} |g| K_N \quad (K_N \text{ positive}) \\
&< \varepsilon' + |g|_{\mathcal{C}^0} \varepsilon' \qquad\qquad\qquad\quad (K_N \text{ normalized, concentrated}) \\
&\leq (1 + 2|f|_{\mathcal{C}^0}) \varepsilon' \\
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Given $\varepsilon > 0$, there exists $N_o(f)$ such that $|f - t_N f|_{\mathcal{C}^0} < \epsilon$ for all $N \geq N_o(f)$

Uniform Féjer polynomial approximation (continued)

For $N \geq N_o(f)$, $(t_N f - f)(y)$ is $(t_N g)(0)$, which is $\langle g, K_N \rangle$,

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&< \varepsilon' + |g|_{\mathcal{C}^0} \varepsilon' \quad (K_N \text{ normalized, concentrated}) \\
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We have close approximation of f by the wrong polynomials in the wrong space

Outline

Mean-square Fourier polynomial approximation is good

Given f in $\mathcal{C}^0(X)$

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Recall

- The Fourier polynomial $s_{2N}f$ is the $\mathcal{L}^2$-best degree-$2N$ trigonometric polynomial approximation of f, by general inner product space principles
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The Fourier polynomials mean-square converge to f because the Féjer polynomials uniformly converge to f

Outline

Zoom in to study pointwise convergence too

Fourier polynomial approximation is preserved under zoom-in

$$(s_N f - f)(y) = (s_N f_y)(0) = \langle f_y, D_N \rangle$$

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> *If f has left and right derivatives at y, not necessarily equal, then $\{(s_N f)(y)\} \xrightarrow{N} f(y)$*