MATHEMATICS 332: ALGEBRA — ASSIGNMENT 4

Reading: Gallian, chapters . . .

Problems:

1. Let G be a group. Let $x \in G$ have order n, and let $y \in G$ have order m, where n and m are relatively prime.

(a) Show that if $xy = yx$ then xy has order nm.

(b) Show that if $xy \neq yx$ then xy need not have order mn.

2. Let G be a group, and let H and K be subgroups. Neither H nor K is assumed to be normal.

(a) Prove that the map

$$h(H \cap K) \longmapsto hK$$

is well defined and gives a bijection from $H/(H \cap K)$ to the set of cosets gK contained in HK.

(b) Now assume that G is finite. Use part (a) to prove the formula

$$|HK| = \frac{|H|\,|K|}{|H \cap K|}.$$

3. Let G be the direct product of two cyclic groups of prime order p. How many subgroups of order p does G have?

4. Let $G = \mathrm{GL}_n(\mathbb{R})$, let $H = \mathrm{SL}_n(\mathbb{R})$, and let $K = \{\lambda I_n : \lambda \in \mathbb{R}^\times\}$. For what n is G the direct product of H and K?

5. Let a group G be the internal semidirect product of a kernel group K and a complementary group H,

$$K \lhd G, \quad H \cap K = 1, \quad HK = G.$$

Prove that every subgroup G_o of G that contains K determines a unique subgroup H_o of H that is complementary to K in G_o, i.e., $H_oK = G_o$.