MATHEMATICS 332: ALGEBRA — EXERCISE ON A RIGHT-ADJOINT

1. Review

Let A be a commutative ring with 1. Every A-module is assumed to have the property that $1_A \cdot x = x$ for all x in the module. As in the discussion of tensors, let B be a second ring-with-identity B and assume that we have a homomorphism

$$\alpha : A \longrightarrow B, \quad 1_A \longmapsto 1_B.$$

Thus B is an A-algebra under the rule

$$a \cdot b = \alpha(a)b, \quad a \in A,\ b \in B,$$

and similarly any B-module N also has the structure of an A-module,

$$a \odot n = \alpha(a) \cdot n, \quad a \in A,\ n \in N,$$

but in practice we drop α from the notation and write ab and an.

The process of viewing every B-module as an A-module and every B-module map as an A-module map, in both cases by forgetting some of the full B-action and instead restricting it to the A-action, is encoded in the *forgetful functor* (or *restriction functor*),

$$\begin{aligned}
\mathrm{Res}^B_A &: \{B\text{-modules}\} \longrightarrow \{A\text{-modules}\},\\
\mathrm{Res}^B_A &: \{B\text{-module maps}\} \longrightarrow \{A\text{-module maps}\}.
\end{aligned}$$

The forgetful functor helps to describe a more interesting *right-adjoint induction functor*,

$$\begin{aligned}
\mathrm{Ind}^B_A &: \{A\text{-modules}\} \longrightarrow \{B\text{-modules}\},\\
\mathrm{Ind}^B_A &: \{A\text{-module maps}\} \longrightarrow \{B\text{-module maps}\}.
\end{aligned}$$

If $g : N \longrightarrow N'$ is an A-module map then its right-adjoint induced map takes the form $\mathrm{Ind}^B_A g : \mathrm{Ind}^B_A N \longrightarrow \mathrm{Ind}^B_A N'$, and if $\tilde{g} : N' \longrightarrow N''$ is a second A-module map then

$$\mathrm{Ind}^B_A(\tilde{g} \circ g) = (\mathrm{Ind}^B_A \tilde{g}) \circ (\mathrm{Ind}^B_A g).$$

The desired properties of right-adjoint induction are as follows.

- The right-adjoint induction functor should be a *right-adjoint* of restriction:
  For every B-module M and A-module N there is an abelian group isomorphism

  $$i_{M,N} : \mathrm{Hom}_B(M, \mathrm{Ind}^B_A N) \xrightarrow{\sim} \mathrm{Hom}_A(\mathrm{Res}^B_A M, N).$$

- The right-adjoint induction functor should be *natural in M*:

For every B-module map $f : M' \longrightarrow M$ and every A-module N, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_B(M, \mathrm{Ind}_A^B N) & \xrightarrow{i_{M,N}} & \mathrm{Hom}_A(\mathrm{Res}_A^B M, N) \\ {\scriptstyle -\circ f}\downarrow & & \downarrow{\scriptstyle -\circ \mathrm{Res}_A^B f} \\ \mathrm{Hom}_B(M', \mathrm{Ind}_A^B N) & \xrightarrow{i_{M',N}} & \mathrm{Hom}_A(\mathrm{Res}_A^B M', N) \end{array}$$

where "$-\circ$" denotes precomposition.

- And the right-adjoint induction functor should be *natural in N*:
  For every B-module M and every A-module map $g : N \longrightarrow N'$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_B(M, \mathrm{Ind}_A^B N) & \xrightarrow{i_{M,N}} & \mathrm{Hom}_A(\mathrm{Res}_A^B M, N) \\ {\scriptstyle \mathrm{Ind}_A^B g\circ -}\downarrow & & \downarrow{\scriptstyle g\circ -} \\ \mathrm{Hom}_B(M, \mathrm{Ind}_A^B N') & \xrightarrow{i_{M,N'}} & \mathrm{Hom}_A(\mathrm{Res}_A^B M, N') \end{array}$$

where "$\circ-$" denotes postcomposition.

Right-adjoint induction is not standard usage.

2. Hom-Group Formation as Right-Adjoint Induction

The right-adjoint construction is a bit trickier than the left-adjoint construction, the tensor product. For objects, the construction is *hom-group formation*,

$$\mathrm{Hom}_A(B, \cdot) : \{A\text{-modules}\} \longrightarrow \{B\text{-modules}\}, \quad N \longmapsto \mathrm{Hom}_A(B, N).$$

Here $\mathrm{Hom}_A(B, N)$ is a B-module under the rule that for any $b \in B$ and any $f \in \mathrm{Hom}_A(B, N)$, the action of s on f is

$$(b \cdot f)(x) = f(xb) \quad \text{for all } x \in B.$$

For maps, the construction is composition. That is, if $g : N \longrightarrow N'$ is an A-module map then its right-adjoint induced map is

$$\mathrm{Ind}_A^B g : \mathrm{Hom}_A(B, N) \longrightarrow \mathrm{Hom}_A(B, N'), \quad f \longmapsto g \circ f.$$

To see that $\mathrm{Ind}_A^B g$ is B-linear, compute that for any $f \in \mathrm{Hom}_A(B, N)$ and any $b, x \in B$,

$$(g \circ (b \cdot f))(x) = g((b \cdot f)(x)) = g(f(xb)) = (g \circ f)(xb) = (b \cdot (g \circ f))(x).$$

That is, $(\mathrm{Ind}_A^B g)(b \cdot f) = s \cdot (\mathrm{Ind}_A^B g)f$.

Theorem 2.1 (Hom-Group Formation is Right-Adjoint Induction)**.** *Let A and B be rings-with-unit, and let $\alpha : A \longrightarrow B$ be a ring homomorphism such that $\alpha(1_A) = 1_B$. Then hom-group formation is a right-adjoint of restriction, is natural in M, and is natural in N.*

Prove the theorem.