MULTILINEAR ALGEBRA: THE TENSOR PRODUCT

This writeup is drawn closely from chapter 27 of Paul Garrett's text **Abstract Algebra**, available from Chapman and Hall/CRC publishers and also available online at Paul Garrett's web site.

Throughout the writeup, let A be a commutative ring with 1. Every A-module is assumed to have the *unital* property that $1_A \cdot x = x$ for all x in the module. Also, the reader is alerted that A-modules are not assumed to be free unless so stated.

1. The Tensor Product: Mapping Property and Uniqueness

Definition 1.1 (Mapping Property of the Tensor Product)**.** *Let M and N be A-modules. Their* **tensor product over** *A is another A-module and a bilinear map from the product of M and N to it,*

$$\tau : M \times N \longrightarrow M \otimes_A N,$$

having the following property: For every A-bilinear map from the product to an A-module,

$$\phi : M \times N \longrightarrow X,$$

there exists a unique A-linear map from the tensor product to the same module,

$$\Phi : M \otimes_A N \longrightarrow X,$$

such that $\Phi \circ \tau = \phi$, i.e., such that the following diagram commutes,

$$\begin{array}{ccc} M \otimes_A N & & \\ {\scriptstyle \tau}\uparrow & \overset{\Phi}{\searrow} & \\ M \times N & \xrightarrow{\ \phi\ } & X. \end{array}$$

That is, the one bilinear map $\tau : M \times N \longrightarrow M \otimes_A N$ reduces all other bilinear maps out of $M \times N$ to linear maps out of $M \otimes_A N$.

Proposition 1.2 (Uniqueness of the Tensor Product)**.** *Let M and N be A-modules. Given two tensor products of M and N over A,*

$$\tau_1 : M \times N \longrightarrow T_1 \quad \textit{and} \quad \tau_2 : M \times N \longrightarrow T_2,$$

there is a unique A-module isomorphism $i : T_1 \longrightarrow T_2$ such that $i \circ \tau_1 = \tau_2$, i.e., such that the following diagram commutes,

$$\begin{array}{ccccc} & & M \times N & & \\ & {\scriptstyle \tau_1}\swarrow & & \searrow{\scriptstyle \tau_2} & \\ T_1 & & \xrightarrow{\ i\ } & & T_2. \end{array}$$

Proof. Since T_1 and T_2 are both tensor products over A, there are unique A-linear maps

$$i : T_1 \longrightarrow T_2 \quad \text{such that } i \circ \tau_1 = \tau_2$$

and

$$j : T_2 \longrightarrow T_1, \quad \text{such that } j \circ \tau_2 = \tau_1.$$

We want to show that i is an isomorphism.

The composition

$$j \circ i : T_1 \longrightarrow T_1$$

is an A-linear map such that

$$(j \circ i) \circ \tau_1 = j \circ (i \circ \tau_1) = j \circ \tau_2 = \tau_1.$$

The definition says that there is a *unique* such A-linear map, and certainly the identity map on T_1 fits the bill. Thus $j \circ i$ is the identity map on T_1. Similarly, $i \circ j$ is the identity map on T_2. □

2. The Tensor Product: Existence

Proposition 2.1 (Existence of the Tensor Product)**.** *Let M and N be A-modules. Then a tensor product $\tau : M \times N \longrightarrow M \otimes_A N$ exists.*

Proof. Let $i : M \times N \longrightarrow F$ be the free A-module on the set $M \times N$. (Note: F is enormous.) While F does have the desired tensor product property of converting maps out of $M \times N$ into linear maps, F is not the tensor product because the map i is not bilinear, is not even a map of algebraic structures. On the other hand, the maps $\phi : M \times N \longrightarrow X$ that F converts to linear maps are completely general, not necessarily bilinear, and so we can collapse the structure of F somewhat and still retain enough of its behavior to convert bilinear maps into linear ones as desired. To collapse F appropriately, let S be its A-submodule generated by its elements that measure the failure of i to be bilinear,

$$\left\{ \begin{array}{l} i(m+m',n) - i(m,n) - i(m',n) \\ i(am,n) - a\,i(m,n) \\ i(m,n+n') - i(m,n) - i(m,n') \\ i(m,an) - a\,i(m,n) \end{array} \right\} \quad \text{where} \quad \left\{ \begin{array}{r} m, m' \in M \\ n, n' \in N \\ a \in A \end{array} \right\}.$$

Form the quotient $Q = F/S$ and take the quotient map,

$$q : F \longrightarrow Q.$$

The composition $q \circ i$ is bilinear since i is bilinear up to S, while q is linear and kills S. For example,

$$\begin{aligned} (q \circ i)(m+m',n) &= q(i(m+m',n)) \\ &= q(i(m,n) + i(m',n) + s) \quad \text{where } s \in S \\ &= q(i(m,n)) + q(i(m',n)) + q(s) \\ &= (q \circ i)(m,n) + (q \circ i)(m',n). \end{aligned}$$

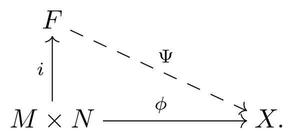
So now, to show that a tensor product is

$$q \circ i : M \times N \longrightarrow Q,$$

we must verify that it uniquely converts bilinear maps out of $M \times N$ to linear maps. So consider any bilinear map of A-modules,

$$\phi : M \times N \longrightarrow X.$$

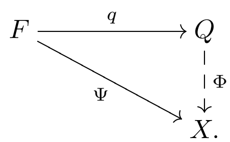
The mapping property of the free module F gives a unique commutative diagram in which the map Ψ is A-linear,



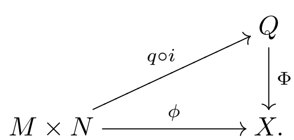
Furthermore, Ψ kills S because the diagram commutes and ϕ is bilinear. For example,

$$\begin{aligned}\Psi\big(i(m+m',n) - i(m,n) - i(m',n)\big) \\ &= (\Psi\circ i)(m+m',n) - (\Psi\circ i)(m,n) - (\Psi\circ i)(m',n) \\ &= \phi(m+m',n) - \phi(m,n) - \phi(m',n) \\ &= 0,\end{aligned}$$

and similarly for the other generators of S as well. Thus Ψ factors through the quotient Q,



The map Φ is linear because Ψ and q are linear and q surjects. (Since Φ is not a composition of the other two maps, this point may deserve a moment's thought.) Concatenate the previous two diagrams and then consolidate to get the desired diagram,



Furthermore, if also $\widetilde{\Phi}\circ q\circ i = \phi$ then $\widetilde{\Phi}\circ q = \Phi\circ q$ by the uniqueness property of the free module, and thus $\widetilde{\Phi} = \Phi$ since q surjects. In sum, the A-module $M\otimes_A N = Q$ and the bilinear map $\tau = q\circ i$ satisfy the tensor product mapping property. □

3. Tangible Descriptions

For any $(m,n)\in M\times N$, the image $\tau(m,n)\in M\otimes_A N$ is denoted $m\otimes n$. Since τ is bilinear, some relations in $M\otimes_A N$ are

$$\begin{aligned}&(m+m')\otimes n = m\otimes n + m'\otimes n,\\ &(am)\otimes n = a(m\otimes n) = m\otimes(an),\\ &m\otimes(n+n') = m\otimes n + m\otimes n',\\ &(a+a')(m\otimes n) = a(m\otimes n) + a'(m\otimes n),\end{aligned}$$

and so on.

As an application of the mapping property, we prove

Proposition 3.1 (Tensor Product Generators)**.** *Let M and N be A-modules. Then the tensor product $\tau : M\times N \longrightarrow M\otimes_A N$ is generated by the monomials $m\otimes n$ where $m\in M$ and $n\in N$. Furthermore, if a set of generators of M over A is $\{m_i\}$*

and a set of generators of N over A is $\{n_j\}$ then a set of generators of $M \otimes_A N$ is $\{m_i \otimes n_j\}$.

Proof. Let $T = M \otimes_A N$, let S be the A-submodule of T generated by the monomials, let $Q = T/S$ be the quotient, and let $q : T \longrightarrow Q$ be the quotient map. Also, let $z : M \times N \longrightarrow Q$ and $Z : T \longrightarrow Q$ be the zero maps. Certainly

$$Z \circ \tau = z,$$

but also, since $\tau(M \times N) \subset S$,

$$q \circ \tau = z.$$

Thus the uniqueness statement in the mapping property of the tensor product gives $q = Z$. In other words, S is all of T.

As for the second statement in the proposition, the first statement shows that any monomial in $M \otimes_A N$ takes the form of the left side of the equality

$$\left(\sum_i a_i m_i\right) \otimes \left(\sum_j \tilde{a}_j n_j\right) = \sum_{i,j} a_i \tilde{a}_j m_i \otimes n_j.$$

That is, the equality shows that any monomial in $M \otimes_A N$ is a linear combination of $\{m_i \otimes n_j\}$. Since any element of $M \otimes_A N$ is a linear combination of monomials in turn, we are done. $\square$

As an example of using the previous proposition, let m and n be positive integers. We will show that

$$\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \approx \mathbb{Z}/g\mathbb{Z} \quad \text{where } g = \gcd(m, n).$$

In particular, and perhaps surprisingly, if $\gcd(m, n) = 1$ then $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is zero.

Indeed, since $\mathbb{Z}/m\mathbb{Z}$ is generated by $1 \bmod m$, and $\mathbb{Z}/n\mathbb{Z}$ is generated by $1 \bmod n$, the proposition says that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is generated in turn by $1 \otimes 1$ (now denoting the cosets by their representatives). Thus $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ is cyclic, i.e., it is a quotient of $\mathbb{Z}$. Next, write

$$g = km + \ell n.$$

Then

$$g(1 \otimes 1) = (km + \ell n)(1 \otimes 1) = km \otimes 1 + 1 \otimes \ell n = 0 \otimes 1 + 1 \otimes 0 = 0.$$

Thus multiplication by g annihilates $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$, making the tensor product a quotient of $\mathbb{Z}/g\mathbb{Z}$. On the other hand, the map

$$\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/g\mathbb{Z}, \quad (x \bmod m, y \bmod n) \longmapsto xy \bmod g$$

is well defined because

$$(x + m\mathbb{Z})(y + n\mathbb{Z}) = xy + xn\mathbb{Z} + ym\mathbb{Z} + mn\mathbb{Z} \subset xy + g\mathbb{Z},$$

and it is bilinear, and it surjects. The mapping property of the tensor product thus gives a surjection from $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/g\mathbb{Z}$.

4. Multiplicativity of Rank

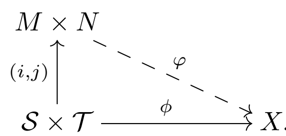
Lemma 4.1. *Let $i : \mathcal{S} \longrightarrow M$ and $j : \mathcal{T} \longrightarrow N$ be free A-modules. For any set map from the product of the sets to an A-module,*

$$\phi : \mathcal{S} \times \mathcal{T} \longrightarrow X,$$

there exists a unique A-bilinear map from the product of the free modules to the same module,

$$\varphi : M \times N \longrightarrow X,$$

such that $\varphi \circ (i, j) = \phi$, i.e., such that the following diagram commutes,



Proof. At most one such φ exists. Indeed, for any $t \in \mathcal{T}$, the condition

$$\varphi(i(s), j(t)) = \phi(s, t) \quad \text{for all } s \in \mathcal{S}$$

determines φ on $M \times \{j(t)\}$ since $i : \mathcal{S} \longrightarrow M$ is free and φ is linear in its first argument. And for any $m \in M$, the values $\varphi(m, j(t))$ as t varies in $\mathcal{T}$ determine φ on $\{m\} \times N$ since $j : \mathcal{T} \longrightarrow N$ is free and φ is linear in its second argument.

As usual, the uniqueness argument determines the construction. For each fixed $t \in \mathcal{T}$, the map $\phi(\cdot, t) : \mathcal{S} \longrightarrow X$ factors through $i : \mathcal{S} \longrightarrow M$,

$$\phi(\cdot, t) = \ell(i(\cdot), j(t)) \quad \text{where } \ell(\cdot, j(t)) : M \longrightarrow X \text{ is linear.}$$

(Note that $\ell(\cdot, n)$ is not defined for general n, only for $n = j(t)$ where $t \in \mathcal{T}$.) For any fixed $m \in M$, the map

$$\ell(m, j(\cdot)) : \mathcal{T} \longrightarrow X$$

factors through $j : \mathcal{T} \longrightarrow N$,

$$\ell(m, j(\cdot)) = \varphi(m, j(\cdot)) \quad \text{where } \varphi(m, \cdot) : N \longrightarrow X \text{ is linear.}$$

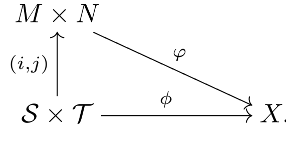
View $\varphi(m, n)$ as a function of its parameter m along with its argument n,

$$\varphi : M \times N \longrightarrow X.$$

Thus

$$\varphi(i(s), j(t)) = \ell(i(s), j(t)) = \phi(s, t) \quad \text{for all } (s, t) \in \mathcal{S} \times \mathcal{T}.$$

In other words, we have a unique commutative diagram



The map φ is linear in its second component, and the relation $\varphi(m, j(t)) = \ell(m, j(t))$ for all m and any given t says that also φ is linear in its first component when its second component takes the form $j(t)$, so in fact φ is linear in its first component overall,

$$
\begin{aligned}
&\varphi(m+m',n) \\
&\quad = \varphi(m+m', \sum_{t\in\mathcal{T}} a_t j(t)) && \text{substituting for } n \\
&\quad = \sum a_t\,\varphi(m+m', j(t)) && \text{since } \varphi(m+m',\cdot) \text{ is linear} \\
&\quad = \sum a_t(\varphi(m,j(t)) + \varphi(m',j(t)) && \text{since } \varphi(\cdot, j(t)) \text{ is linear} \\
&\quad = \sum a_t\varphi(m,j(t)) + \sum a_t\varphi(m',j(t)) && \text{by the distributive law in } X \\
&\quad = \varphi(m, \sum a_t j(t)) + \varphi(m', \sum a_t j(t)) && \text{since } \varphi(m,\cdot) \text{ and } \varphi(m',\cdot) \text{ are linear} \\
&\quad = \varphi(m,n) + \varphi(m',n) && \text{substituting } n.
\end{aligned}
$$

□

Proposition 4.2 (Multiplicativity of Rank). *Let $i : \mathcal{S} \longrightarrow M$ and $j : \mathcal{T} \longrightarrow N$ be free A-modules. Let $\tau : M \times N \longrightarrow M \otimes_A N$ be the tensor product map. Then*

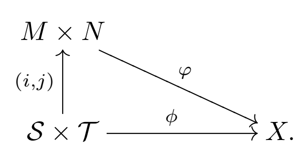
$$\tau \circ (i,j) : \mathcal{S} \times \mathcal{T} \longrightarrow M \otimes_A N, \quad (s,t) \longmapsto i(s) \otimes j(t)$$

is again a free A-module. Thus

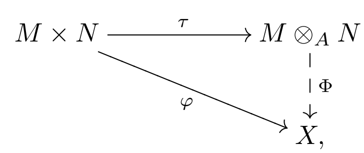
$$\text{rank}_A(M \otimes_A N) = \text{rank}_A(M) \cdot \text{rank}_A(N).$$

Remark: In contrast to the initial construction of the tensor product from a free module, this proposition constructs a free module from a tensor product. The encapsulation $\text{free}(\mathcal{S} \times \mathcal{T}) = \text{free}(\mathcal{S}) \otimes_A \text{free}(\mathcal{T})$ of the proposition is in contrast to the result $\text{free}(\mathcal{S} \sqcup \mathcal{T}) = \text{free}(\mathcal{S}) \times \text{free}(\mathcal{T})$ from the writeup on free modules.

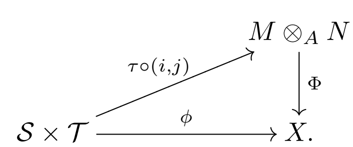
Proof. Let X be an A-module, and consider a set map $\phi : \mathcal{S} \times \mathcal{T} \longrightarrow X$. By the lemma, there exists a unique bilinear map $\varphi : M \times N \longrightarrow X$ such that the following diagram commutes,



Now the tensor product mapping property gives a commutative diagram



where Φ is linear. Concatenate with the previous diagram to get the desired commutativity property of Φ,



To show that any linear $\Phi : M \otimes_A N \longrightarrow X$ such that $\Phi \circ \tau \circ (i, j) = \phi$ is unique, note that $\varphi = \Phi \circ \tau : M \times N \longrightarrow X$ is bilinear and satisfies $\varphi \circ (i, j) = \phi$. By the lemma φ is unique, and then also Φ is unique by nature of the tensor product. $\square$

5. Invariance of Rank

Proposition 5.1 (Invariance of Rank Under Change of Ring)**.** *Let $A \subset B$ be a containment of rings with $1_A = 1_B$. Let $i : \mathcal{S} \longrightarrow F$ be a free A-module, and let $\tau : B \times F \longrightarrow B \otimes_A F$ be the tensor product. Then*

$$k : \mathcal{S} \longrightarrow B \otimes_A F, \quad s \longmapsto \tau(1_B, i(s)) = 1_B \otimes i(s)$$

is a free B-module of the same rank. Here the B-module structure of $B \otimes_A F$ is

$$b(b' \otimes m) = (bb') \otimes m.$$

Remark: The proposition and others like it hold if rather than the containment $A \subset B$ we have a homomorphism $\alpha : A \longrightarrow B$ of rings-with-unit with $\alpha(1_A) = 1_B$. We will discuss this issue later in the handout.

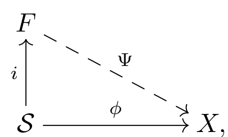
Remark: This proposition again constructs a free module from a tensor product. This time the encapsulation might be $\text{free}_B(\mathcal{S}) = B \otimes_A \text{free}_A(\mathcal{S})$.

Proof. Consider any set-map from $\mathcal{S}$ to a B-module,

$$\phi : \mathcal{S} \longrightarrow X.$$

We want the usual diagram involving a B-linear map $\Phi : B \otimes_A F \longrightarrow X$.

View X as an A-module. Because $i : \mathcal{S} \longrightarrow F$ is a free A-module, we have a diagram



where Ψ is A-linear. To incorporate the domain of the A-bilinear map $\tau : B \times F \longrightarrow B \otimes_A F$ into the diagram, introduce a map that makes reference to the B-module structure of X,

$$\Gamma : B \times F \longrightarrow X, \quad \Gamma(b, m) = b\Psi(m).$$

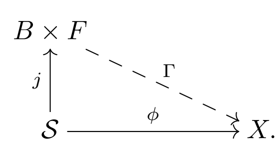
To see that Γ is A-bilinear, compute that for $a \in A$ and $b, b' \in B$ and $m, m' \in F$,

$$\begin{aligned}
\Gamma(b + b', m) &= (b + b')\Psi(m) = b\Psi(m) + b'\Psi(m) = \Gamma(b, m) + \Gamma(b', m), \\
\Gamma(ab, m) &= ab\Psi(m) = a\Gamma(b, m), \\
\Gamma(b, m + m') &= b\Psi(m + m') = b\Psi(m) + b\Psi(m') = \Gamma(b, m) + \Gamma(b, m'), \\
\Gamma(b, am) &= b\Psi(am) = ab\Psi(m) = a\Gamma(b, m).
\end{aligned}$$

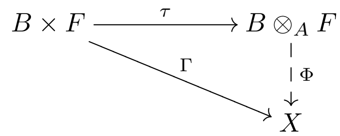
Also introduce the map

$$j : \mathcal{S} \longrightarrow B \times F, \quad s \longmapsto (1_B, i(s)).$$

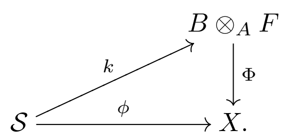
Now a diagram that essentially repeats the previous one, but with the desired product at the upper left corner, commutes as well,



The mapping property of the tensor product gives a commutative diagram



where Φ is A-linear. And so concatenating the diagrams gives (recalling the map k from the statement of the proposition)



We need to show that Φ is B-linear. Recall from the statement of the proposition that the B-module structure of $B \otimes_A F$ is $b(b' \otimes m) = (bb') \otimes m$. Now compute,

$$\begin{aligned}
\Phi(b(b' \otimes m)) &= \Phi((bb') \otimes m) && \text{by the structure of } B \otimes_A F \\
&= \Gamma(bb', m) && \text{because } \Gamma = \Phi \circ \tau \\
&= (bb')\Psi(m) && \text{by definition of } \Gamma \\
&= b(b'\Psi(m)) && \text{by the structure of } B \otimes_A F \\
&= b\Gamma(b', m) && \text{by definition of } \Gamma \\
&= b\Phi(b' \otimes m) && \text{because } \Gamma = \Phi \circ \tau.
\end{aligned}$$

Finally, we need to show that Φ is unique. Given a set-map $\phi : \mathcal{S} \longrightarrow X$ where X is a B-module, and given a B-linear map $\Phi : B \otimes_A F \longrightarrow X$ such that $\Phi \circ \tau \circ (1_B \times i) = \phi$, let $\Gamma = \Phi \circ \tau : B \times F \longrightarrow X$. Then Γ is B-linear in its first argument. For example, remembering that $b \otimes m$ denotes $\tau(b, m)$,

$$\Gamma(b\tilde{b}, m) = \Phi(b\tilde{b} \otimes m) = \Phi(b(\tilde{b} \otimes m)) = b\Phi(\tilde{b} \otimes m) = b\Gamma(\tilde{b}, m).$$

Similarly Γ is A-linear in its second argument. The values

$$\Gamma(1_B, i(s)) = \Phi(1_B \otimes i(s)) = \phi(s)$$

are determined by ϕ. Consequently, so is Γ overall in consequence of the linearity of Γ in each of its arguments,

$$\Gamma(b, \sum a_s i(s)) = \sum a_s b \Gamma(1_B, i(s)).$$

Because Γ is A-bilinear, the mapping property of the tensor product $B \otimes_A F$ shows that the B-map Φ compatible with Γ is unique. In sum, the map Γ determined by any Φ compatible with ϕ is unique to ϕ, and Φ is unique to Γ, altogether making Φ unique to ϕ. $\square$

Note that

$$B \otimes_A F = \{b(1 \otimes m) : b \in B, m \in F\},$$

and the algebra of $B \otimes_A F$ involves rules such as

$$\begin{aligned}
&b((1 \otimes m) + (1 \otimes m')) = b(1 \otimes m) + b(1 \otimes m'),\\
&(b + b')(1 \otimes m) = b(1 \otimes m) + b'(1 \otimes m),\\
&(bb')(1 \otimes m) = b(b(1 \otimes m)),\\
&b(1 \otimes m) = 1 \otimes bm \text{ if and only if } b \in A.
\end{aligned}$$

That is, if we think of F as containing an independent copy $Ai(s)$ of A for each element s of the generating set $\mathcal{S}$, then correspondingly $B \otimes_A F$ contains an independent copy $B(1 \otimes i(s))$ of B for each s.

As an example, the complex number system

$$\mathbb{C} = \mathbb{R} \times \mathbb{R}i = \{x + yi : x, y \in \mathbb{R}\}$$

is a free $\mathbb{R}$-module of rank 2. Naïvely converting it to a $\mathbb{C}$-module gives

$$\{z + wi : z, w \in \mathbb{C}\} = \mathbb{C},$$

a free $\mathbb{C}$-module whose rank is only 1 due to dependence among the generators once the ring of scalars is enlarged. However, converting it to a $\mathbb{C}$-module via the tensor product gives

$$\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R} \times \mathbb{R}i) = \mathbb{C}(1 \otimes 1) \times \mathbb{C}(1 \otimes i),$$

a free rank-2 $\mathbb{C}$-module that acquires its rank and its basis naturally from the original rank-2 $\mathbb{R}$-module, with no accidental collapsing.

Similarly, consider a field $F = \mathbb{Q}(\alpha)$ where α is algebraic over $\mathbb{Q}$. Let $f(X) \in \mathbb{Q}[X]$ be the minimal monic polynomial over $\mathbb{Q}$ satisfied by α. Thus f is irreducible over $\mathbb{Q}$ and

$$F \cong \mathbb{Q}[X]/\langle f(X) \rangle.$$

As a polynomial over $\mathbb{R}$ (rather than over $\mathbb{Q}$), f has real roots and pairs of complex conjugate roots, which is to say that it factors into a product of linear polynomials and irreducible quadratic polynomials,

$$f(X) = \prod_{i=1}^{r} L_i(X) \prod_{j=1}^{s} Q_j(X) \quad \text{in } \mathbb{R}[X].$$

The tensor product $\mathbb{R} \otimes_{\mathbb{Q}} F$ decomposes correspondingly,

$$\mathbb{R} \otimes_{\mathbb{Q}} F \cong \mathbb{R}[X]/\langle f(X) \rangle \cong \prod_{i} \mathbb{R}[X]/\langle L_i(X) \rangle \prod_{j} \mathbb{R}[X]/\langle Q_j(X) \rangle \cong \mathbb{R}^r \times \mathbb{C}^s.$$

(For example, $\mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{2}) \cong \mathbb{R}^2$ whereas $\mathbb{R}(\sqrt{2}) = \mathbb{R}$.) The natural map $F \longrightarrow \mathbb{R} \otimes_{\mathbb{Q}} F$ thus gives rise to the so-called *canonical embedding* $F \longrightarrow \mathbb{R}^r \times \mathbb{C}^s$ of algebraic number theory.

6. The Tensor Product of Maps

Consider two A-linear maps,

$$f : M \longrightarrow M', \qquad g : N \longrightarrow N'.$$

The map

$$\tau' \circ (f \times g) : M \times N \longrightarrow M' \otimes_A N', \quad (m, n) \longmapsto f(m) \otimes g(n)$$

is readily seen to be bilinear. For example,

$$f(m + m') \otimes g(n) = (f(m) + f(m')) \otimes g(n) = f(m) \otimes g(n) + f(m') \otimes g(n).$$

The mapping property of $M \otimes_A N$ gives a unique linear map,

$$\begin{array}{ccc} M \otimes_A N & \xrightarrow{f\otimes g} & M' \otimes_A N' \\ \uparrow \tau & & \uparrow \tau' \\ M \times N & \xrightarrow{f\times g} & M' \times N'. \end{array}$$

In symbols, the formula for $f \otimes g$ is

$$(f \otimes g)(m \otimes n) = f(m) \otimes g(n).$$

7. Tensor Product Formation as Left-Adjoint Induction

We have been working with a ring-with-identity A. Introduce now a second ring-with-identity B and a homomorphism

$$\alpha : A \longrightarrow B, \quad 1_A \longmapsto 1_B.$$

Thus B is an A-algebra under the rule

$$a \cdot b = \alpha(a)b, \quad a \in A,\ b \in B.$$

And similarly any B-module N also has the structure of an A-module,

$$a \odot n = \alpha(a) \cdot n, \quad a \in A,\ n \in N.$$

In practice we drop α from the notation and write ab and an, tacitly understanding that the previous two displays are really what is meant. Especially, when A is a subring of B (with $1_A = 1_B$), the inclusion map α is naturally omitted. In other contexts, e.g., $\alpha : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$, a bit more information is left out of the notation when we drop α, but the resulting gain in tidiness is worthwhile.

Although we may view every B-module as an A-module, and every B-module map as an A-module map, in both cases by forgetting some of the full B-action and instead restricting it to the A-action, strictly speaking a B-module viewed as an A-module is not the same algebraic structure as the original B-module. That is, we have a *forgetful functor* or *restriction functor*,

$$\begin{aligned} \mathrm{Res}_A^B &: \{B\text{-modules}\} \longrightarrow \{A\text{-modules}\}, \\ \mathrm{Res}_A^B &: \{B\text{-module maps}\} \longrightarrow \{A\text{-module maps}\}. \end{aligned}$$

Although the restriction functor does nothing, in the sense that

$$\begin{aligned} &\mathrm{Res}_A^B N = N \text{ as an abelian group for all } B\text{-modules } N, \\ &\mathrm{Res}_A^B g = g \text{ as an abelian group map for all } B\text{-module maps } g, \end{aligned}$$

still $\mathrm{Res}_A^B N$ emphatically does not fully equal N since they are algebraic structures of different types, and similarly $\mathrm{Res}_A^B g$ does not fully equal g. Still a person could easily wonder whether there is any point to the forgetful functor.

There is. Its use is to help describe a more interesting *left-adjoint induction functor*,

$$\begin{aligned} \mathrm{Ind}_A^B &: \{A\text{-modules}\} \longrightarrow \{B\text{-modules}\}, \\ \mathrm{Ind}_A^B &: \{A\text{-module maps}\} \longrightarrow \{B\text{-module maps}\}. \end{aligned}$$

Here if $f : M \longrightarrow M'$ is an A-module map then its left-adjoint induced B-module map is a map between the induced B-modules,

$$\mathrm{Ind}_A^B f : \mathrm{Ind}_A^B M \longrightarrow \mathrm{Ind}_A^B M',$$

and if $\tilde{f} : M' \longrightarrow M''$ is a second A-module map then the induced B-module map of the composition is the composition of the induced B-module maps,

$$\mathrm{Ind}_A^B(\tilde{f} \circ f) = (\mathrm{Ind}_A^B \tilde{f}) \circ (\mathrm{Ind}_A^B f).$$

We want three properties to hold for the left-adjoint induction functor.

- The left-adjoint induction functor should be a *left-adjoint* of restriction:
  For every A-module M and B-module N there is an abelian group isomorphism
  $$i_{M,N} : \mathrm{Hom}_B(\mathrm{Ind}_A^B M, N) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Res}_A^B N).$$
- The left-adjoint induction functor should be *natural in M*:
  For every A-module map $f : M' \longrightarrow M$ and every B-module N, there is a commutative diagram
  $$\begin{array}{ccc} \mathrm{Hom}_B(\mathrm{Ind}_A^B M, N) & \xrightarrow{i_{M,N}} & \mathrm{Hom}_A(M, \mathrm{Res}_A^B N) \\ {\scriptstyle -\circ \mathrm{Ind}_A^B f}\downarrow & & \downarrow{\scriptstyle -\circ f} \\ \mathrm{Hom}_B(\mathrm{Ind}_A^B M', N) & \xrightarrow{i_{M',N}} & \mathrm{Hom}_A(M', \mathrm{Res}_A^B N) \end{array}$$
  where "$-\circ$" denotes precomposition.
- And the left-adjoint induction functor should be *natural in N*:
  For every A-module M and every B-module map $g : N \longrightarrow N'$, there is a commutative diagram
  $$\begin{array}{ccc} \mathrm{Hom}_B(\mathrm{Ind}_A^B M, N) & \xrightarrow{i_{M,N}} & \mathrm{Hom}_A(M, \mathrm{Res}_A^B N) \\ {\scriptstyle g\circ -}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_A^B g\circ -} \\ \mathrm{Hom}_B(\mathrm{Ind}_A^B M, N') & \xrightarrow{i_{M,N'}} & \mathrm{Hom}_A(M, \mathrm{Res}_A^B N') \end{array}$$
  where "$\circ-$" denotes postcomposition.

The reader is warned that *left-adjoint induction* is absolutely **not** standard usage.

Theorem 7.1 (Tensor Product Formation is Left-Adjoint Induction)**.** *Let A and B be rings-with-unit, and let $\alpha : A \longrightarrow B$ be a ring homomorphism such that $\alpha(1_A) = 1_B$. The tensor product*

$$\begin{aligned} B \otimes_A \cdot :& \{A\text{-modules}\} \longrightarrow \{B\text{-modules}\}, & M &\longmapsto B \otimes_A M, \\ \mathrm{id}_B \otimes_A \cdot :& \{A\text{-module maps}\} \longrightarrow \{B\text{-module maps}\}, & f &\longmapsto \mathrm{id}_B \otimes_A f \end{aligned}$$

is a left-adjoint of restriction, is natural in M, and is natural in N.

Proof. Define

$$i_{M,N} : \mathrm{Hom}_B(B \otimes_A M, N) \longrightarrow \mathrm{Hom}_A(M, \mathrm{Res}_A^B N)$$

by the formula

$$(i_{M,N}\Phi)(m) = \Phi(1 \otimes m), \quad m \in M,$$

and define

$$j_{M,N} : \mathrm{Hom}_A(M, \mathrm{Res}_A^B N) \longrightarrow \mathrm{Hom}_B(B \otimes_A M, N)$$

by the formula

$$(j_{M,N}\phi)(b \otimes m) = b\,\phi(m), \quad b \in B,\ m \in M.$$

Here the product $b\,\phi(m)$ uses the original structure of N as a B-module, even though not all of that structure is used in our understanding of $\mathrm{Res}_A^B N$ as an A-module. Then i and j are readily seen to be abelian group homomorphisms, and (all the symbols meaning what they must)

$$(j(i\Phi))(b \otimes m) = b\,(i\Phi)(m) = b\,\Phi(1 \otimes m) = \Phi(b \otimes m)$$

while

$$(i(j\phi))(m) = (j\phi)(1 \otimes m) = 1 \cdot \phi(m) = \phi(m).$$

Thus $i_{M,N}$ is an isomorphism.

For naturality in M, compute that for every A-module map $f : M' \longrightarrow M$ and every B-module map $\Phi : B \otimes_A M \longrightarrow N$, for any $m' \in M'$,

$$\begin{aligned}(i_{M',N}(\Phi \circ (\mathrm{id}_B \otimes f)))(m') &= (\Phi \circ (\mathrm{id}_B \otimes f))(1 \otimes m') \\ &= \Phi(1 \otimes f(m')) \\ &= (i_{M,N}\Phi)(f(m')) \\ &= ((i_{M,N}\Phi) \circ f)(m').\end{aligned}$$

Thus $i_{M',N}(\Phi \circ (\mathrm{id}_B \otimes f)) = (i_{M,N}\Phi) \circ f$.

For naturality in N, compute that for every B-module map $\Phi : B \otimes_A M \longrightarrow N$ and every B module map $g : N \longrightarrow N'$, for any $m \in M$,

$$\begin{aligned}(i_{M,N'}(g \circ \Phi))(m) &= (g \circ \Phi)(1 \otimes m) \\ &= g(\Phi(1 \otimes m)) \\ &= \mathrm{Res}_A^B g((i_{M,N}\Phi)(m)) \\ &= (\mathrm{Res}_A^B g \circ (i_{M,N}\Phi))(m).\end{aligned}$$

Thus $i_{M,N'}(g \circ \Phi) = \mathrm{Res}_A^B g \circ (i_{M,N}\Phi)$. □

For an example, let k be a field and V a vector space over k. Let K be a superfield of k. Proposition 5.1 says that

$$\dim_K(K \otimes_k V) = \dim_k(V),$$

and even that:

If $\{e_j\}$ is a basis of V over k then $\{1_K \otimes e_j\}$ is a basis of $K \otimes_k V$ over K.

Furthermore, we have seen that for every vector space W over K,

$$\mathrm{Hom}_K(K \otimes_k V, W) \approx \mathrm{Hom}_k(V, \mathrm{Res}_k^K(W)).$$

Thus this special case of left-adjoint induction is understandably referred to as *extension of scalars.*

8. The Multiple Tensor Product

Let $n \geq 2$, and let $M_1, \cdots, M_n$ be A-modules. The n-fold tensor product

$$\tau : M_1 \times \cdots \times M_n \longrightarrow M_1 \otimes_A \cdots \otimes_A M_n \qquad \text{where } \tau \text{ is } n\text{-linear}$$

can be characterized by a mapping property summarized in a diagram similar to Definition 1.1,

$$\begin{array}{ccc} M_1 \otimes_A \cdots \otimes_A M_n & & \\ \uparrow{\scriptstyle \tau} & \overset{\Phi}{\searrow} & \\ M_1 \times \cdots \times M_n & \xrightarrow{\phi} & X, \end{array}$$

in which ϕ is any n-linear A-map and Φ is linear.

The n-fold tensor product can be constructed as before. Specifically, consider the free A-module

$$i : M_1 \times \cdots \times M_n \longrightarrow F,$$

and let S be the A-submodule of F generated by all elements of the form

$$\begin{aligned} &i(\cdots, m_i + m_i', \cdots) - i(\cdots, m_i, \cdots) - i(\cdots, m_i', \cdots), \\ &i(\cdots, am_i, \cdots) - a\, i(\cdots, m_i, \cdots), \end{aligned}$$

Consider the quotient $Q = F/S$ and the quotient map $q : F \longrightarrow Q$. Then the n-fold tensor product is

$$q \circ i : M_1 \times \cdots \times M_n \longrightarrow Q.$$

The n-fold tensor product of maps is formed similarly to the binary tensor product of maps as well,

$$(f_1 \otimes \cdots \otimes f_n)(m_1 \otimes \cdots \otimes m_n) = f_1(m_1) \otimes \cdots \otimes f_n(m_n).$$

It is a matter of routine to verify that there is a natural isomorphism

$$(M_1 \otimes_A M_2) \otimes_A M_3 \xrightarrow{\sim} M_1 \otimes_A M_2 \otimes_A M_3$$

under which for all A-linear maps

$$f_i : M_i \longrightarrow M_i', \quad i = 1, 2, 3,$$

there is a commutative diagram

$$\begin{array}{ccc} (M_1 \otimes_A M_2) \otimes_A M_3 & \longrightarrow & M_1 \otimes_A M_2 \otimes_A M_3 \\ {\scriptstyle (f_1 \otimes_A f_2) \otimes_A f_3} \downarrow & & \downarrow {\scriptstyle f_1 \otimes_A f_2 \otimes_A f_3} \\ (M_1' \otimes_A M_2') \otimes_A M_3' & \longrightarrow & M_1' \otimes_A M_2' \otimes_A M_3'. \end{array}$$

And similarly for $M_1 \otimes_A (M_2 \otimes_A M_3)$. That is, the formation of tensor products is associative.