MATHEMATICS 332: ALGEBRA — EXERCISE ON MODULES OVER A PID

Reading: Class handout on modules over a PID

Problem:

Let A be PID. (So, for example, we might have $A = \mathbb{Z}$ or $A = k[X]$ where k is a field.) Let $n \geq 2$ be an integer. Consider any primitive vector in $A^{\oplus n}$,

$$v = (a_1, \cdots, a_n) \in A^{\oplus n}, \quad \gcd(a_1, \cdots, a_n) = 1.$$

This exercise shows conceptually that v is the first column of an n-by-n matrix M with entries in A having determinant 1.

(a) Consider a nonzero vector $v' = (a'_1, \cdots, a'_n) \in A^{\oplus n}$. Show that if $av' \in Av$ for some nonzero $a \in A$ then $v' \in Av$.

(b) Consider a free A-module and a submodule,

$$F = A^{\oplus n}, \quad S = Av.$$

Use (a) to show that the quotient F/S is torsion-free, meaning that if $q \in F/S$ and $a \in A$ is nonzero and $aq = 0_{F/S}$ then $q = 0_{F/S}$. (To connect your reasoning clearly to (a), write elements of F/S as cosets, or, better yet, use the natural projection map $\pi : F \longrightarrow F/S$.)

(c) By the structure theorem for finitely generated modules over a PID, we have a basis $(f_1, \cdots, f_n)$ of F and a nonzero ideal $\mathfrak{a}_1 \subset A$ such that

$$F = Af_1 \oplus \cdots \oplus Af_n, \quad S = \mathfrak{a}_1 f_1, \quad F/S = (A/\mathfrak{a}_1)f_1 \oplus Af_2 \oplus \cdots \oplus Af_n.$$

And $\mathfrak{a}_1$ anniliates $A/\mathfrak{a}_1$. Combine (b) with the environment described here to explain—with no further reference to elements—why $\mathfrak{a}_1 = A$. Consequently, we may take $f_1 = v$.

(d) Thus $A^{\oplus n}$ has a basis $(v, f_2, \cdots, f_n)$. And so, letting $(e_1, \cdots, e_n)$ denote the standard basis, the A-linear map

$$T : A^{\oplus n} \longrightarrow A^{\oplus n}, \quad e_1 \mapsto v,\ e_2 \mapsto f_2,\ \cdots,\ e_n \mapsto f_n$$

is an isomorphism, so that consequently $\det T \in A^\times$. What is the matrix M of T with respect to the standard basis? Explain why scaling the last column of M by $(\det T)^{-1}$ finishes the problem. (Here is where the condition $n \geq 2$ matters: we are not modifying the given vector v.)