MATHEMATICS 332: ALGEBRA – EXERCISE ON RINGS AND IDEALS

Consider the interval $[0, 1]$ of the real number line. Consider the set R of continuous real-valued functions whose domain is $[0, 1]$,

$$R = \mathcal{C}([0, 1], \mathbb{R}) = \{f : [0, 1] \longrightarrow \mathbb{R} \,\big|\, f \text{ is continuous}\}.$$

Define ring operations on R in the natural way, meaning that for all $f, g \in R$ the sum $f + g$ and the product fg are taken pointwise,

$$\begin{aligned} (f + g)(x) &= f(x) + g(x), \quad && x \in [0, 1], \\ (fg)(x) &= f(x)g(x), && x \in [0, 1]. \end{aligned}$$

Because the sum and product of continuous functions are continuous, indeed $f + g$ and fg are again elements of R.

(a) Let I be a proper ideal of R, i.e., I is not all of R. Show that there exists a point $x \in [0, 1]$ such that $f(x) = 0$ for all $f \in I$.

(b) Part (a) shows that the only possible maximal ideals of R are the sets of elements of R that vanish at some particular point $x \in [0, 1]$,

$$I_x = \{f \in R : f(x) = 0\}.$$

Show that each such ideal I_x is indeed maximal.