# GROUP PRODUCTS

Many beginning group theory texts distinguish between the *external* direct product and the *internal* direct product of groups. This writeup explains a viewpoint from which there is literally no difference between them. The idea is to define the product by its *characterizing mapping property*, describing how it interacts with other groups, rather than by its internal details.

## 1. Definition of the Product via a Mapping Property

**Definition 1.1** (Product of Two Groups)**.** *Let $G_1$ and $G_2$ be groups. A* **product** *of $G_1$ and $G_2$ is*

- *a group $G$*
- *and homomorphisms $\pi_i : G \longrightarrow G_i$ for $i = 1, 2$ (called* **projections***),*

*which we may view as the configuration*

$$\begin{tikzcd} & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G_1 & & G_2, \end{tikzcd}$$

*having the following property: For any group $\widetilde{G}$ and homomorphisms $f_i : \widetilde{G} \longrightarrow G_i$ for $i = 1, 2$,*

$$\begin{tikzcd} & \widetilde{G} \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\ & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G_1 & & G_2, \end{tikzcd}$$

*there exists a unique homomorphism $f : \widetilde{G} \longrightarrow G$ to make the resulting diagram commute,*

$$\begin{tikzcd} & \widetilde{G} \arrow[d, dashed, "f"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\ & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G_1 & & G_2. \end{tikzcd}$$

In natural language, the definition says that

> *Any collection of homomorphisms from a group into the productands factor uniquely through the product.*

## 2. Uniqueness

The mapping property definition of the product shows immediately that there can be essentially only one such thing.

**Proposition 2.1** (Uniqueness of the Product)**.** *Let $G_1$ and $G_2$ be groups, and let $(G, \pi_1, \pi_2)$ and $(G', \pi_1', \pi_2')$ both be products of $G_1$ and $G_2$. Then there is a unique isomorphism from $G$ to $G'$.*

*Proof.* First we consider endomorphisms $\epsilon$ of $G$ that make the following diagram commute:

$$\begin{tikzcd} & G \arrow[d, "\epsilon"] \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] & \\ & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G_1 & & G_2. \end{tikzcd}$$

Clearly the identity endomorphism $\mathrm{id}_G : G \longrightarrow G$ works. Furthermore, the mapping property characterization of $G$ as a product of $G_1$ and $G_2$ shows that the identity endomorphism is the *only* endomorphism $\epsilon$ of $G$ that works. The same observation applies to $G'$, of course.

Now, since $G'$ is a product of $G_1$ and $G_2$, the diagram

$$\begin{tikzcd} & G \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] & \\ & G' \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\ G_1 & & G_2 \end{tikzcd}$$

is uniquely completed,

$$\begin{tikzcd} & G \arrow[d, dashed, "f"] \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] & \\ & G' \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\ G_1 & & G_2. \end{tikzcd}$$

And since $G$ is a product of $G_1$ and $G_2$, the diagram

$$\begin{tikzcd} & G' \arrow[ddl, bend right, "\pi_1'"'] \arrow[ddr, bend left, "\pi_2'"] & \\ & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G_1 & & G_2 \end{tikzcd}$$

is uniquely completed,

$$\begin{tikzcd}
& G' \arrow[d, dashed, "f'"] \arrow[ddl, bend right, "\pi_1'"'] \arrow[ddr, bend left, "\pi_2'"] & \\
& G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G_1 & & G_2.
\end{tikzcd}$$

Concatenate the completed diagrams in two ways to get two more diagrams,

$$\begin{tikzcd}
& G \arrow[d, "f' \circ f"] \arrow[ddl, bend right, "\pi_1"'] \arrow[ddr, bend left, "\pi_2"] & \\
& G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G_1 & & G_2
\end{tikzcd}
\qquad
\begin{tikzcd}
& G' \arrow[d, "f \circ f'"] \arrow[ddl, bend right, "\pi_1'"'] \arrow[ddr, bend left, "\pi_2'"] & \\
& G' \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\
G_1 & & G_2.
\end{tikzcd}$$

As observed earlier in argument, it follows that $f' \circ f = \mathrm{id}_G$ and $f \circ f' = \mathrm{id}_{G'}$. Thus $f$ and $f'$ are isomorphisms. $\square$

## 3. Existence

We don't yet know that a product of two groups $G_1$ and $G_2$ exists at all. One construction of a product is indeed the cartesian product,

$$G = G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1,\ g_2 \in G_2\},$$

with the group operation defined componentwise in terms of the given group operations,

$$(g_1, g_2) \circ_G (g_1', g_2') = (g_1 \circ_{G_1} g_1',\ g_2 \circ_{G_2} g_2').$$

The projections are what they must be,

$$\begin{aligned}
\pi_1 : G &\longrightarrow G_1, \quad \pi_1(g_1, g_2) = g_1, \\
\pi_2 : G &\longrightarrow G_2, \quad \pi_2(g_1, g_2) = g_2.
\end{aligned}$$

To verify the mapping property, suppose that we are given any group $\widetilde{G}$ along with homomorphisms

$$f_i : \widetilde{G} \longrightarrow G_i, \quad i = 1, 2.$$

Then the only set-map $f : \widetilde{G} \longrightarrow G$ that makes the diagram

$$\begin{tikzcd}
& \widetilde{G} \arrow[d, "f"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\
& G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
G_1 & & G_2
\end{tikzcd}$$

commute is the map

$$f : \tilde{G} \longrightarrow G, \quad f(\tilde{g}) = (f_1(\tilde{g}), f_2(\tilde{g})),$$

and this map is indeed a homomorphism.

The maps

$$\iota_1 : G_1 \longrightarrow G, \quad g_1 \longmapsto (g_1, e_2),$$
$$\iota_2 : G_2 \longrightarrow G, \quad g_2 \longmapsto (e_1, g_2)$$

are monomorphisms. Thus the cartesian product $G$ contains an isomorphic copy $G_1 \times \{e_2\}$ of $G_1$ and and isomorphic copy $\{e_1\} \times G_2$ of $G_2$ as subgroups, but $G_1$ and $G_2$ are not literally subgroups of $G$.

The cartesian product $G = G_1 \times G_2$ is the *external direct product* of $G_1$ and $G_2$.

## 4. Another Manifestation of the Product

Suppose now that $G_1$ and $G_2$ are subgroups of some group $G$ and that furthermore,

$$\begin{aligned}
&G_1G_2 = G, \text{ i.e., } G = \{g_1g_2 : g_1 \in G_1, g_2 \in G_2\}\\
&G_1 \cap G_2 = \{e_G\},\\
&G_1 \lhd G \text{ and } G_2 \lhd G.
\end{aligned}$$

Then for all $g_1 \in G_1$ and $g_2 \in G_2$,

$$g_1g_2g_1^{-1}g_2^{-1} = \begin{cases} g_1g_1' = g_1'' & \text{for some } g_1'', \text{ since } G_1 \lhd G, \\ g_2'g_2^{-1} = g_2'' & \text{for some } g_2'', \text{ since } G_2 \lhd G. \end{cases}$$

And so $g_1g_2g_1^{-1}g_2^{-1} \in G_1 \cap G_2 = \{e\}$. That is, each of $G_1$, $G_2$ centralizes the other,

$$g_1g_2 = g_2g_1 \text{ for all } g_1 \in G_1 \text{ and } g_2 \in G_2.$$

Thus the maps

$$\pi_1 : G \longrightarrow G_1, \quad \pi_1(g_1g_2) = g_1,$$
$$\pi_2 : G \longrightarrow G_2, \quad \pi_2(g_1g_2) = g_2$$

are homomorphisms, and given any group $\widetilde{G}$ and homomorphisms

$$f_i : \widetilde{G} \longrightarrow G_i, \quad i = 1, 2,$$

the map

$$f : \widetilde{G} \longrightarrow G, \quad f(\widetilde{g}) = f_1(\widetilde{g})f_2(\widetilde{g})$$

is the unique homomorphism that gives the suitable commutative diagram. In sum, $G$ satisfies the characterizing mapping property of to be a product of $G_1$ and $G_2$.

A group $G$ of the form described in this section is the *internal direct product* of $G_1$ and $G_2$.

We know from earlier in the handout that an internal direct product $G$ of $G_1$ and $G_2$ must be uniquely isomorphic to the cartesian product $G_1 \times G_2$. Of course, the unique isomorphism is clear in any case,

$$g_1g_2 \longmapsto (g_1, g_2).$$

## 5. The Product of an Arbitrary Collection of Groups

Using the mapping-theoretic notion of product, the generalization from two groups to an arbitrary collection of groups is effortless.

**Definition 5.1** (Product of a Collection of Groups)**.** *Let $A$ be an arbitrary index set, and consider a collection of groups indexed by $A$,*

$$\{G_\alpha : \alpha \in A\}.$$

*A* **product** *of the groups $\{G_\alpha\}$ is*

- *a group $G$*
- *and homomorphisms $\pi_\alpha : G \longrightarrow G_\alpha$ for $\alpha \in A$ (called* **projections***),*

*having the following property: For any group $\widetilde{G}$ and homomorphisms $f_\alpha : \widetilde{G} \longrightarrow G_\alpha$ for $\alpha \in A$, there exists a unique homomorphism $f : \widetilde{G} \longrightarrow G$ such that*

$$\pi_\alpha \circ f = f_\alpha, \quad \alpha \in A.$$

As before, it follows immediately that any two products of the groups $\{G_\alpha\}$ are uniquely isomorphic.

Checking whether the constructions of external and internal products for an arbitrary collection of groups both work requires a bit of care when the index set $A$ is infinite. The details are omitted here. The salient point is that working with the mapping-theoretic definition requires no distinction between the twofold case and the arbitrary case.

More importantly, one can also use the same mapping property to describe the product in other *categories*, e.g., in the environment where the sets are topological spaces rather than groups and the arrows are continuous maps rather than homomorphisms. Uniqueness of the product is always immediate as above (our uniqueness argument made no use of the particulars of groups or homomorphisms), but addressing existence is a category-specific matter.

One can also define a **coproduct** in any category by reversing the arrows in the product definition. The interested reader might try to construct a coproduct of two groups. An obstacle does arise.