# THE THREE GROUP ISOMORPHISM THEOREMS

## 1. The First Isomorphism Theorem

**Theorem 1.1** (An image is a natural quotient)**.** *Let*

$$f : G \longrightarrow \widetilde{G}$$

*be a group homomorphism. Let its kernel and image be*

$$K = \ker(f), \qquad \widetilde{H} = \operatorname{im}(f),$$

*respectively a normal subgroup of $G$ and a subgroup of $\widetilde{G}$. Then there is a natural isomorphism*

$$\tilde{f} : G/K \xrightarrow{\sim} \tilde{H}, \quad gK \longmapsto f(g).$$

*Proof.* The map $\tilde{f}$ is well defined because if $g'K = gK$ then $g' = gk$ for some $k \in K$ and so

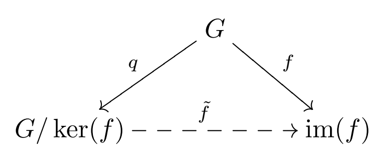
$$f(g') = f(gk) = f(g)f(k) = f(g)\tilde{e} = f(g).$$

The map $\tilde{f}$ is a homomorphism because $f$ is a homomorphism,

$$\begin{aligned} \tilde{f}(gK\, g'K) &= \tilde{f}(gg'K) && \text{by definition of coset multiplication} \\ &= f(gg') && \text{by definition of } \tilde{f} \\ &= f(g)f(g') && \text{because } f \text{ is a homomorphism} \\ &= \tilde{f}(gK)\tilde{f}(g'K) && \text{by definition of } \tilde{f}. \end{aligned}$$

To show that $\tilde{f}$ injects, it suffices to show that $\ker(\tilde{f})$ is only the trivial element $K$ of $G/K$. Compute that if $\tilde{f}(gK) = \tilde{e}$ then $f(g) = \tilde{e}$, and so $g \in K$, making $gK = K$ as desired. The map $\tilde{f}$ surjects because $\widetilde{H} = \operatorname{im}(f)$. $\square$

A diagrammatic display of the theorem that captures its idea that an image is isomorphic to a quotient is as follows:




For a familiar example of the theorem, let

$$T : V \longrightarrow W$$

be a linear transformation. The theorem says that there is a resulting natural isomorphism

$$\widetilde{T} : V/\text{nullspace}(T) \xrightarrow{\sim} \text{range}(T).$$

The quotient vector space $V/\text{nullspace}(T)$ is the set of translates of the nullspace. If we expand a basis of the nullspace,

$$\{v_1, \cdots, v_\nu\} \quad (\text{where } \nu \text{ is the nullity of } T),$$

to a basis of $V$,

$$\{v_1, \cdots, v_\nu, v_{\nu+1}, \cdots, v_n\},$$

then a basis of the quotient (now denoting the nullspace $N$ for brevity) consists of the cosets

$$\{v_{\nu+1} + N, \cdots, v_n + N\},$$

Thus the isomorphism $V/N \xrightarrow{\sim} T(V)$ encompasses the basic result from linear algebra that the rank of $T$ and the nullity of $T$ sum to the dimension of $V$. The dimension of the original codomain $W$ is irrelevant here.

Often the First Isomorphism Theorem is applied in situations where the original homomorphism is an epimorphism $f : G \longrightarrow \widetilde{G}$. The theorem then says that consequently the induced map $\tilde{f} : G/K \longrightarrow \widetilde{G}$ is an isomorphism. For example,

- Since every cyclic group is by definition a homomorphic image of $\mathbb{Z}$, and since the nontrivial subgroups of $\mathbb{Z}$ take the form $n\mathbb{Z}$ where $n \in \mathbb{Z}_{>0}$, we see clearly now that every cyclic group is either

  $$G \approx \mathbb{Z} \quad \text{or} \quad G \approx \mathbb{Z}/n\mathbb{Z}.$$

  Consider a finite cyclic group,

  $$G = \langle g \rangle, \quad \pi : \mathbb{Z} \longrightarrow G, \quad \pi(1) = g, \quad \ker(\pi) = n\mathbb{Z}.$$

  Consider also a subgroup,

  $$H = \langle g^k \rangle.$$

  Then $\pi^{-1}(H) = k\mathbb{Z}$, so that

  $$H \approx k\mathbb{Z}/(k\mathbb{Z} \cap n\mathbb{Z}) = k\mathbb{Z}/\mathrm{lcm}(k,n)\mathbb{Z}.$$

  The multiply-by-$k$ map followed by a natural quotient map gives an epimorphsim $\mathbb{Z} \longrightarrow k\mathbb{Z}/\mathrm{lcm}(k,n)\mathbb{Z}$, and the kernel of the composition is $(\mathrm{lcm}(k,n)/k)\mathbb{Z} = (n/\gcd(k,n))\mathbb{Z}$. Thus

  $$H \approx \mathbb{Z}/(n/\gcd(k,n))\mathbb{Z}.$$

  Hence the subgroup $H = \langle g^k \rangle$ of the order-$n$ cyclic group $G = \langle g \rangle$ has order

  $$|\langle g^k \rangle| = n/\gcd(k,n).$$

  Especially, $H$ is all of $G$ when $\gcd(k,n) = 1$, and so $G$ has $\varphi(n)$ generators.
- The epimorphism $|\ | : \mathbb{C}^\times \longrightarrow \mathbb{R}^+$ has as its kernel the complex unit circle, denoted $\mathbb{T}$,

  $$\mathbb{T} = \{z \in \mathbb{C}^\times : |z| = 1\}.$$

  The quotient group $\mathbb{C}^\times/\mathbb{T}$ is the set of circles in $\mathbb{C}$ centered at the origin and having positive radius, with the multiplication of two such circles returning the circle whose radius is the product of the radii. The isomorphism

  $$\mathbb{C}^\times/\mathbb{T} \xrightarrow{\sim} \mathbb{R}^+$$

  takes each circle to its radius.
- The epimorphism $\exp : \mathbb{C} \longrightarrow \mathbb{C}^\times$ has as its kernel a dilated vertical copy of the integers,

  $$K = 2\pi i\mathbb{Z}.$$

Each element of the quotient group $\mathbb{C}/2\pi i\mathbb{Z}$ is a translate of the kernel. The quotient group overall can be viewed as the strip of complex numbers with imaginary part between 0 and $2\pi$, rolled up into a tube. The isomorphism

$$\mathbb{C}/2\pi i\mathbb{Z} \xrightarrow{\sim} \mathbb{C}^\times$$

takes each horizontal line at height $y$ to the ray making angle $y$ with the positive real axis. Loosely, the exponential maps shows us a view of the tube looking "down" it from the end.

- The epimorphism $\det : \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathbb{R}^\times$ has as its kernel the special linear group $\mathrm{SL}_n(\mathbb{R})$. Each element of the quotient group $\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R})$ is the equivalence class of all $n$-by-$n$ real matrices having a given nonzero determinant. The isomorphism

$$\mathrm{GL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{R}) \xrightarrow{\sim} \mathbb{R}^\times$$

takes each equivalence class to the shared determinant of all its members.

- The epimorphism $\mathrm{sgn} : S_n \longrightarrow \{\pm 1\}$ has as its kernel the alternating group $A_n$. The quotient group $S_n/A_n$ can be viewed as the set

$$\{\mathrm{even}, \mathrm{odd}\},$$

forming the group of order 2 having *even* as the identity element. The isomorphism

$$S_n/A_n \xrightarrow{\sim} \{\pm 1\}$$

takes *even* to 1 and *odd* to $-1$.

## 2. The Second Isomorphism Theorem

**Theorem 2.1.** *Let $G$ be a group. Let $H$ be a subgroup of $G$ and let $K$ be a normal subgroup of $G$. Then there is a natural isomorphism*

$$HK/K \xrightarrow{\sim} H/(H \cap K), \quad hK \longmapsto h(H \cap K).$$

*Proof.* Routine verifications show that $HK$ is a group having $K$ as a normal subgroup and that $H \cap K$ is a normal subgroup of $H$. The map

$$H \longrightarrow HK/K, \quad h \longmapsto hK$$

is a surjective homomorphism having kernel $H \cap K$, and so the first theorem gives an isomorphism

$$H/(H \cap K) \xrightarrow{\sim} HK/K, \quad h(H \cap K) \longmapsto hK.$$

The desired isomorphism is the inverse of the isomorphism in the display. □

Before continuing, it deserves quick mention that if $G$ is a group and $H$ is a subgroup and $K$ is a normal subgroup then $HK = KH$. Indeed, because $K$ is normal,

$$HK = \{hK : h \in H\} = \{Kh : h \in H\} = KH.$$

We will cite this little fact later in the writeup.

As an example of the second ismorphism theorem, consider a general linear group, its special linear subgroup, and its center,

$$G = \mathrm{GL}_2(\mathbb{C}), \qquad H = \mathrm{SL}_2(\mathbb{C}), \qquad K = \mathbb{C}^\times I_2.$$

Then

$$HK = G, \qquad H \cap K = \{\pm I_2\}.$$

The isomorphism given by the theorem is therefore

$$\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times I_2 \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{C})/\{\pm I_2\}, \quad \mathbb{C}^\times m \longmapsto \{\pm m\}.$$

The groups on the two sides of the isomorphism are the *projective* general and special linear groups. Even though the general linear group is larger than the special linear group, the difference disappears after projectivizing,

$$\mathrm{PGL}_2(\mathbb{C}) \xrightarrow{\sim} \mathrm{PSL}_2(\mathbb{C}).$$

## 3. The Third Isomorphism Theorem

**Theorem 3.1** (Absorption property of quotients)**.** *Let $G$ be a group. Let $K$ be a normal subgroup of $G$, and let $N$ be a subgroup of $K$ that is also a normal subgroup of $G$. Then*

$$K/N \text{ is a normal subgroup of } G/N,$$

*and there is a natural isomorphism*

$$(G/N)/(K/N) \xrightarrow{\sim} G/K, \quad gN\cdot(K/N) \longmapsto gK.$$

*Proof.* The map

$$G/N \longrightarrow G/K, \quad gN \longmapsto gK$$

is well defined because if $g'N = gN$ then $g' = gn$ for some $n \in N$ and so because $N \subset K$ we have $g'K = gK$. The map is a homomorphism because

$$gN\,g'N = gg'N \longmapsto gg'K = gK\,g'K.$$

The map clearly surjects. Its kernel is $K/N$, showing that $K/N$ is a normal subgroup of $G/N$, and the first theorem gives an isomorphism

$$(G/N)/(K/N) \xrightarrow{\sim} G/K, \quad gN\cdot(K/N) \longmapsto gK,$$

as claimed. □

For example, let $n$ and $m$ be positive integers with $n \mid m$. Thus

$$m\mathbb{Z} \subset n\mathbb{Z} \subset \mathbb{Z}$$

and all subgroups are normal since $\mathbb{Z}$ is abelian. The third isomorphism theorem gives the isomorphism

$$(\mathbb{Z}/m\mathbb{Z})/(n\mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}/n\mathbb{Z}, \quad (k+m\mathbb{Z})+n\mathbb{Z} \longmapsto k+n\mathbb{Z}.$$

And so the following diagram commutes because both ways around are simply $k \mapsto k+n\mathbb{Z}$:

$$\begin{array}{ccccc} & & \mathbb{Z} & & \\ & \swarrow & & \searrow & \\ \mathbb{Z}/m\mathbb{Z} & \longrightarrow & (\mathbb{Z}/m\mathbb{Z})/(n\mathbb{Z}/m\mathbb{Z}) & \longrightarrow & \mathbb{Z}/n\mathbb{Z}. \end{array}$$

In words, if one reduces modulo $m$ and then further reduces modulo $n$, then the second reduction subsumes the first.

## 4. Preliminary Lemma

**Lemma 4.1.** *Let $f : G \longrightarrow \widetilde{G}$ be an epimorphism, and let $K$ be its kernel. Then there is a bijective correspondence*

$$\{\textit{subgroups of } G \textit{ containing } K\} \longleftrightarrow \{\textit{subgroups of } \widetilde{G}\}$$

*given by*

$$H \longrightarrow f(H),$$
$$f^{-1}(\widetilde{H}) \longleftarrow \widetilde{H}.$$

*And the bijection restricts to*

$$\{\textit{normal subgroups of } G \textit{ containing } K\} \longleftrightarrow \{\textit{normal subgroups of } \widetilde{G}\}.$$

*Proof.* If $H$ is a subgroup of $G$ containing $K$ then $f(H)$ is a subgroup of $\widetilde{G}$, and

$$f^{-1}(f(H)) = \{g \in G : f(g) \in f(H)\} \supset H.$$

To show equality, note that if for any $g \in G$,

$$\begin{aligned} f(g) \in f(H) &\implies f(g) = f(h) && \text{for some } h \in H \\ &\implies f(h^{-1}g) = \tilde{e} \\ &\implies h^{-1}g \in K \\ &\implies g \in hK \subset HK = H && \text{since } H \text{ contains } K. \end{aligned}$$

On the other hand, if $\widetilde{H}$ is a subgroup of $\widetilde{G}$ then $f^{-1}(\widetilde{H})$ is a subgroup of $G$ containing $K$. The containment $f(f^{-1}(\widetilde{H})) \subset \widetilde{H}$ is clear, and the containment is equality because $f$ is an epimorphism.

Now suppose that $H$ is a normal subgroup of $G$ containing $K$. Since $f$ is an epimorphism, any $\tilde{g} \in \widetilde{G}$ takes the form $f(g)$, and so

$$\tilde{g}f(H)\tilde{g}^{-1} = f(g)f(H)f(g^{-1}) = f(gHg^{-1}) = f(H),$$

showing that $f(H)$ is a normal subgroup of $\widetilde{G}$. Conversely, suppose that $\widetilde{H}$ is a normal subgroup of $\widetilde{G}$. Then for any $g \in G$,

$$f(gf^{-1}(\widetilde{H})g^{-1}) = f(g)f(f^{-1}(\widetilde{H}))f(g)^{-1} = f(g)\widetilde{H}f(g)^{-1} = \widetilde{H},$$

and so $gf^{-1}(\widetilde{H})g^{-1} = f^{-1}(\widetilde{H})$, showing that $f^{-1}(\widetilde{H})$ is a normal subgroup of $G$, □

As a particular case of the lemma, if $G$ is a group and $K$ is a normal subgroup and $Q = G/K$, then since the natural projection $G \longrightarrow Q$ is an epimorphism, the subgroups of $G$ containing $K$ are in bijective correspondence with the the subgroups of $Q$, and the correspondence preserves normality.

## 5. Solvable Groups

**Definition 5.1.** *A finite group $G$ is* solvable *if there is a series*

$$1 = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$$

*where each quotient $G_i/G_{i-1}$ for $i \in \{1, \cdots, n\}$ is cyclic.*

**Theorem 5.2.** *Let $G$ be a finite group. If $G$ is solvable then any subgroup of $G$ and any quotient group of $G$ are solvable. Conversely, if $K$ is a normal subgroup of $G$ and $Q = G/K$, and $K$ and $Q$ are solvable, then $G$ is solvable.*

*Proof.* Suppose that $G$ is solvable. Let $H$ be any subgroup of $G$, not necessarily normal. Define

$$H_i = H \cap G_i, \quad i \in \{0, \cdots, n\}.$$

Then for any $i \in \{1, \cdots, n\}$ and any $h_i \in H_i$ we have, since $H$ is a group and $G_{i-1} \lhd G_i$,

$$h_i H_{i-1} h_i^{-1} = h_i (H \cap G_{i-1}) h_i^{-1} \subset H \cap G_{i-1} = H_{i-1}.$$

That is, each $H_{i-1}$ is normal in $H_i$,

$$1 = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_{n-1} \lhd H_n = H.$$

The quotients from this series are

$$H_i/H_{i-1} = (H \cap G_i)/(H \cap G_{i-1}).$$

Apply the second isomorphism theorem, substituting

$$G_i \text{ for } G, \quad H \cap G_i \text{ for } H, \quad G_{i-1} \text{ for } K,$$

and the result is

$$H_i/H_{i-1} \xrightarrow{\sim} (H \cap G_i)G_{i-1}/G_{i-1}.$$

Since $(H \cap G_i)G_{i-1}$ is a subgroup of $G_i$ containing $G_{i-1}$, the quotient is a subgroup of $G_i/G_{i-1}$ by the lemma. Any subgroup of a cyclic group is again cyclic, and so $H$ is solvable.

Still assuming that $G$ is solvable, let $K$ be any normal subgroup of $G$. For any $i \in \{1, \cdots, n\}$, since $G_{i-1} \lhd G_i$ and $K \lhd G_i$ we have for any $g_i \in G_i$,

$$g_i G_{i-1} K = G_{i-1} g_i K = G_{i-1} K g_i,$$

and also, as discussed immediately after the second isomorphism theorem, we have $G_{i-1}K = KG_{i-1}$, showing that $K$ normalizes $G_{i-1}K$. In sum, $G_{i-1}K \lhd G_iK$. Also, the natural map

$$G_i \longrightarrow G_iK/G_{i-1}K$$

surjects and is trivial on $G_{i-1}$, and so it factors through the quotient, still surjecting,

$$G_i/G_{i-1} \longrightarrow G_iK/G_{i-1}K.$$

Now define

$$Q_i = G_iK/K, \quad i \in \{0, \cdots, n\}.$$

By the third isomorphism theorem, each $Q_{i-1}$ is normal in $Q_i$,

$$1 = Q_0 \lhd Q_1 \lhd Q_2 \lhd \cdots \lhd Q_{n-1} \lhd Q_n = Q.$$

The quotients from this series are, by the third isomorphism theorem,

$$Q_i/Q_{i-1} = (G_iK/K)/(G_{i-1}K/K) \xrightarrow{\sim} G_iK/G_{i-1}K.$$

Thus $Q_i/Q_{i-1}$ is an image of the cyclic group $G_i/G_{i-1}$. Any image of a cyclic group is again cyclic, and so $Q$ is solvable.

No longer assuming that $G$ is solvable, let $K$ be a normal subgroup of $G$, let $Q = G/K$, and suppose that $K$ and $Q$ are solvable. Then we have a chain

$$1 = K_0 \lhd K_1 \lhd K_2 \lhd \cdots \lhd K_{m-1} \lhd K_m = K$$

with cyclic quotients $K_i/K_{i-1}$, and we have a chain

$$1 = Q_m \lhd Q_{m+1} \lhd \cdots \lhd Q_{n-1} \lhd Q_n = Q,$$

again with cyclic quotients. By the lemma, the second chain gives rise to a chain in $G$,

$$K = G_m \lhd G_{m+1} \lhd \cdots \lhd G_{n-1} \lhd G_n = G.$$

The quotients from this series are, by the third isomorphism theorem,

$$G_i/G_{i-1} \xrightarrow{\sim} (G_i/K)/(G_{i-1}/K) = Q_i/Q_{i-1},$$

which are cyclic, and so the proof is complete. □

There are tidier ways to establish Theorem 5.2. Here we did so using almost no tools in order to showcase the isomorphism theorems.