COSETS IN LAGRANGE'S THEOREM AND IN GROUP ACTIONS

1. Lagrange's Theorem

Let G be a group and H a subgroup, not necessarily normal.

Definition 1.1 (Left H-equivalence)**.** *Two group elements $g, g' \in G$ are* **left H-equivalent** *if they produce the same left coset of H,*

$$g \sim_L g' \quad \text{if } gH = g'H.$$

The verification that left H-equivalence is indeed an equivalence relation on G is straightforward. Thus left H-equivalence partitions G into disjoint equivalence classes, the left cosets,

$$G = \bigsqcup gH \quad \text{(disjoint union of cosets, not union over all } g \in G\text{)}.$$

The **left coset space** is the set of left cosets,

$$G/H = \{gH\} \quad \text{(each element of the set is itself a coset)}.$$

Also, one shows instantly that

$$g \sim_L g' \iff g^{-1}g' \in H, \quad g, g' \in G.$$

Naturally we could also define right H-equivalence and repeat the ideas,

$$G = \bigsqcup Hg, \qquad H\backslash G = \{Hg\}, \qquad g \sim_R g' \iff g'g^{-1} \in H.$$

Now, for any $g \in G$ we have a bijection between H and gH,

$$H \longleftrightarrow gH, \quad h \longleftrightarrow gh.$$

And similarly for H and right cosets Hg. Consequently all cosets have the same cardinality,

$$|gH| = |Hg| = |H|, \quad g \in G.$$

From the decompositions $G = \bigsqcup gH = \bigsqcup Hg$ we then get

$$|G| = |G/H| \cdot |H| = |H\backslash G| \cdot |H|.$$

Define the **index** of H in G to be the shared cardinality of the coset spaces,

$$[G : H] = |G/H| = |H\backslash G|.$$

If G is finite then $[G : H]$ is a positive integer. But by the previous-but-first display,

$$[G : H] = |G|/|H|.$$

And thus:

Theorem 1.2 (Lagrange)**.** *Let G be a finite group, and let H be a subgroup of G. Then $|H|$ divides $|G|$.*

Lagrange's Theorem has many corollaries:

- If G is a prime-order group then it is cyclic.
- if G is a finite group and $a \in G$ then $|a| \mid |G|$.

- (Euler) Let $n \in \mathbb{Z}_{>0}$. Then $a^{\varphi(n)} = 1 \bmod n$ if $\gcd(a, n) = 1$.
- (Fermat) Let p be prime. Then $a^{p-1} = 1 \bmod p$ if $p \nmid a$.

2. Multiplicity of Indices

Let A be a supergroup of B, in turn a supergroup of C,

$$C \subset B \subset A.$$

Thus

$$A = \bigsqcup_i a_i B, \quad [A : B] = |\{a_i\}|$$

and

$$B = \bigsqcup_j b_j C, \quad [B : C] = |\{b_j\}|.$$

Essentially immediately,

$$A = \bigsqcup_{i,j} a_i b_j C.$$

Indeed, the union in the previous display is disjoint because if $a_i' b_j' C = a_i b_j C$ then the cosets $a_i' B$ and $a_i B$ are nondisjoint, making them equal, so that $a_i' = a_i$, and then we have $b_j' C = b_j C$, giving $b_j' = b_j$. Since the union is disjoint and the (i, j)th coset contains the product $a_i b_j$, no two such products are equal unless they involve the same a_i and the same b_j. The multiplicativity of indices follows,

$$[A : C] = |\{a_i b_j\}| = |\{a_i\}|\,|\{b_j\}| = [A : B]\,[B : C].$$

3. Cosets and Actions

Consider a *transitive* action

$$G \times S \longrightarrow S.$$

Here G is a group, S is a set, and the action takes any point of S to any other.

Fix a point $x \in S$, and let G_x be its isotropy subgroup,

$$G_x = \{g \in G : gx = x\}.$$

As we have discussed, G_x is indeed a subgroup of G, but it need not be normal.

There is a natural set bijection between the resulting left coset space and the set,

$$G/G_x \longleftrightarrow S, \quad gG_x \longleftrightarrow gx.$$

To see this, recall that G/G_x is the disjoint union of the left cosets,

$$G = \bigsqcup gG_x,$$

and for any $g, g' \in G$,

$$g'G_x = gG_x \iff g^{-1}g' \in G_x \iff g^{-1}g'x = x \iff g'x = gx.$$

That is, each coset collectively moves x to a well-defined point of S, and distinct cosets move x to distinct points.

For an example, let

$$G = \mathrm{SL}_2(\mathbb{R}),$$
$$S = \mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

Then G acts on S by the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(z) = \frac{az+b}{cz+d}.$$

One key fact here is that

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}\right)(z) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}(z)\right),$$

and another is that

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\operatorname{Im}(z)}{|cz+d|^2}\,.$$

Now take our particular point to be

$$x = i.$$

Then its isotropy groups is the 2-by-2 special orthogonal group,

$$\begin{aligned} G_x &= \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}) : \begin{bmatrix} a & b \\ c & d \end{bmatrix}(i) = i\right\} \\ &= \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})\right\} \\ &= \mathrm{SO}(2). \end{aligned}$$

Thus the complex upper half plane has a completely real group-theoretic description as a coset space,

$$\mathcal{H} \approx \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2).$$

No claim is being made here that the quotient space carries a group structure.