GROUP ACTIONS

1. Review of Homomorphisms

Recall that if $(G, \circ_G)$ and $(\widetilde{G}, \circ_{\widetilde{G}})$ are groups then a set-map

$$f : G \longrightarrow \widetilde{G}$$

is a *homomorphism* if the following diagram commutes:

$$\begin{array}{ccc} G \times G & \xrightarrow{(f,f)} & \widetilde{G} \times \widetilde{G} \\ \circ_G \big\downarrow & & \big\downarrow \circ_{\widetilde{G}} \\ G & \xrightarrow{\quad f \quad} & \widetilde{G}. \end{array}$$

That is, the map f must satisfy the condition

$$f(g \circ_G g') = f(g) \circ_{\widetilde{G}} f(g'), \quad g, g' \in G.$$

An injective homomorphism is a *monomorphism*. A surjective homomorphism is an *epimorphism*. A bijective homomorphism is an *isomorphism*. A homomorphism from a group back to itself is an *endomorphism* and an isomorphism from a group back to itself is an *automorphism*.

Immediately in consequence of the definition, any homomorphism satisfies

$$\begin{aligned} f(e_G) &= e_{\widetilde{G}}, \\ f(g^{-1}) &= (f(g))^{-1} \quad \text{for all } g \in G, \end{aligned}$$

and

f is a monomorphism if and only if its kernel is trivial.

Also, we showed in class that the inverse map of an isomorphism is again an isomorphism. That is, if a bijective set-map between groups preserves algebra then so does its inverse.

And the subgroup test quickly shows that for any homomorphism,

- $\ker(f)$ is a subgroup of G.
- $\mathrm{im}(f)$ is a subgroup of $\widetilde{G}$.
- $f^{-1}(\widetilde{H})$ is a subgroup of G for any subgroup $\widetilde{H}$ of $\widetilde{G}$.

If G is abelian then so is any homomorphic image $f(G)$.

2. Group Actions

Recall also that if G is a group and S is a set then an *action* of G on S is a map

$$G \times S \longrightarrow S, \quad (g, s) \longmapsto gs$$

such that

$$\begin{aligned} es &= s && \text{for all } s \in S, \\ (gg')s &= g(g's) && \text{for all } g, g' \in G,\ s \in S. \end{aligned}$$

The formula $(gg')s = g(g's)$ (called the *associativity* rule for the action) features one group product and three group actions. The associativity rule shows immediately that

$$g(g^{-1}s) = s, \quad g \in G,\ s \in S.$$

3. Isotropy

For any group action, for any element s of the set acted on, the subset of G that fixes s,

$$G_s = \{g \in G : gs = s\},$$

is the *isotropy subgroup* of s. Verifying that G_s is indeed a subgroup is straightforward, using the last display of the previous paragraph.

4. Application of Isotropy: Centralizing Subgroups

Especially, any group G acts on its own power set $\mathcal{P}(G)$ in two ways:

- By *left-translation*, $(g, S) \mapsto gS = \{gs : s \in S\}$.
- By *left-conjugation*, $(g, S) \mapsto gSg^{-1} = \{gsg^{-1} : s \in S\}$.

For any cardinal number k, the two actions restrict to actions of G on the set of cardinality-k subsets of G. In particular, when $k = 1$ they restrict to actions of G on itself. The conjugation action also restricts to the set of sub*groups* of G, and to the set of cardinality-k subgroups of G for any k.

The *centralizer* of any group element $\tilde{g}$ is defined as an isotropy subgroup,

$$Z(\tilde{g}) = \tilde{g}\text{-isotropy under the conjugation action of } G \text{ on itself},$$

That is, the centralizer of $\tilde{g}$ is the subgroup of group elements that commute with $\tilde{g}$,

$$Z(\tilde{g}) = \{g \in G : g\tilde{g} = \tilde{g}g\}.$$

The centralizer of $\tilde{g}$ is a supergroup of the subgroup of G generated by $\tilde{g}$.

For any subset S of G, the centralizer of S is the subgroup of group elements that commute with S,

$$Z(S) = \bigcap_{\tilde{g} \in S} Z(\tilde{g}) = \{g \in G : g\tilde{g} = \tilde{g}g \text{ for all } \tilde{g} \in S\}.$$

The centralizer of S need not contain S. In particular the *center* of the group is the subgroup of elements that commute with the entire group,

$$Z(G) = \{g \in G : g\tilde{g} = \tilde{g}g \text{ for all } \tilde{g} \in G\}.$$

A group may have abelian subgroups that are not central, since *central* connotes commuting with the entire group.

5. Application of Isotropy: Normalizing Subgroups

The *normalizer* of any subset S of G is its isotropy subgroup under the action of G on its power set,

$$N(S) = \{g \in G : gSg^{-1} = S\}.$$

Elements of $N(S)$ need not fix S *pointwise* under conjugation. Conjugation by elements of $N(S)$ may permute S, but it may not move elements out of S.

Especially, for any homomorphism $f : G \longrightarrow \widetilde{G}$,

$$N(\ker(f)) = G.$$

Indeed, for any $k \in \ker(f)$ and any $g \in G$,

$$f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)e_{\widetilde{G}}(f(g))^{-1} = e_{\widetilde{G}}.$$

For another example in the same spirit, consider some subgroups of $\mathrm{GL}_2(F)$ where F is any field,

$$P = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \right\} \quad \text{(the } \textit{parabolic} \text{ subgroup)},$$

$$M = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right\} \quad \text{(the } \textit{maximal Levi component}\text{)},$$

$$N = \left\{ \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \right\} \quad \text{(the } \textit{unipotent radical}\text{)}.$$

The calculation

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

shows that

$$P = MN = NM.$$

Also, the intersection $M \cap N$ is trivial. And, although M and N do not commute, M normalizes N,

$$mnm^{-1} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{bmatrix} = \begin{bmatrix} 1 & abd^{-1} \\ 0 & 1 \end{bmatrix} \in N.$$

That is, the normalizer of N in P is all of P. On the other hand, one can check that N does not normalize M.