MEROMORPHIC CONTINUATION AND FUNCTIONAL EQUATION OF RIEMANN ZETA

The Riemann zeta function is *initially* defined as a sum,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1.$$

This writeup gives Riemann's argument that the closely related function

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1$$

has a meromorphic continuation to the full s-plane, analytic except for simple poles at $s = 0$ and $s = 1$, and the continuation satisfies the functional equation

$$\xi(s) = \xi(1 - s), \quad s \in \mathbb{C}.$$

The continuation is no longer defined by the sum.

The functional equation says that the behavior of the extended $\xi(s)$ on the open left half plane $\{\mathrm{Re}\,(s) < 0\}$ mirrors its reflection on the open right half plane $\{\mathrm{Re}\,(s) > 1\}$, where the original definition of $\zeta(s)$ as a sum is in effect. That is, $\xi(s)$ on the open left half plane gives nothing new. What is truly new is the behavior of the extended $\xi(s)$ on the *critical strip* $\{0 \leq \mathrm{Re}\,(s) \leq 1\}$. Riemann showed that the zeros of $\xi(s)$ in the critical strip give significant information about the distribution of prime numbers.

The Riemann zeta function generalizes vastly, as do all the ideas of this writeup. The resulting extensions ubiquitously encode arithmetic information in their behavior on a critical strip and/or on the critical strip boundary.

Fourier transform. The space of measurable and absolutely integrable functions on $\mathbb{R}$ is

$$\mathcal{L}^1(\mathbb{R}) = \{\text{measurable } f : \mathbb{R} \longrightarrow \mathbb{C} \ : \ \int_{x\in\mathbb{R}} |f(x)|dx < \infty\}.$$

Any $f \in \mathcal{L}^1(\mathbb{R})$ has a *Fourier transform* $\hat{f} : \mathbb{R} \longrightarrow \mathbb{C}$ given by

$$\hat{f}(x) = \int_{y\in\mathbb{R}} f(y)e^{-2\pi iyx}dy.$$

Although the Fourier transform is continuous, it need not belong to $\mathcal{L}^1(\mathbb{R})$. But if also $\int_{x\in\mathbb{R}} |f(x)|^2dx < \infty$ then $\int_{x\in\mathbb{R}} |\hat{f}(x)|^2dx < \infty$.

The Gaussian; Fourier transform of the Gaussian and its dilations. Let $f \in \mathcal{L}^1(\mathbb{R})$ be the *Gaussian function*,

$$f(x) = e^{-\pi x^2}.$$

We need its Fourier transform. Compute that

$$\hat{f}(x) = \int_{y=-\infty}^{\infty} e^{-\pi(y^2+2iyx-x^2)}e^{-\pi x^2}dy = e^{-\pi x^2}\int_{y=-\infty}^{\infty} e^{-\pi(y+ix)^2}dy.$$

Complex contour integration shows that the integral is just the Gaussian integral $\int_{-\infty}^{\infty} e^{-\pi y^2} dy$, and this is 1. Thus

$$\hat{f} = f \quad \text{for the Gaussian } f.$$

Also, for any function $h \in \mathcal{L}^1(\mathbb{R})$ and any positive number r, the Fourier transform of the dilated function $h_r(x) = h(xr)$ is $\widehat{h_r}(x) = r^{-1}\hat{h}(xr^{-1})$. So in particular

$$\text{the Fourier transform of } f(xt^{1/2}) \text{ is } t^{-1/2} f(xt^{-1/2}), \quad t > 0.$$

The theta function; its expression as a sum of dilated Gaussians. Let $\mathcal{H}$ denote the complex upper half plane. The *theta function* on $\mathcal{H}$ is

$$\vartheta : \mathcal{H} \longrightarrow \mathbb{C}, \qquad \vartheta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}.$$

The sum converges very rapidly away from the real axis, making absolute and uniform convergence on compact subsets of $\mathcal{H}$ easy to show, and thus defining a holomorphic function. Specialize to $\tau = it$ with $t > 0$ and again let f be the Gaussian. The theta function along the positive imaginary axis is a sum of dilated Gaussians,

$$\vartheta(it) = \sum_{n \in \mathbb{Z}} f(nt^{1/2}), \quad t > 0.$$

This is a sum of quickly decreasing functions whose graphs narrow as n grows.

Poisson summation; the transformation law of the theta function. For any function $h \in \mathcal{L}^1(\mathbb{R})$ such that the sum $\sum_{d \in \mathbb{Z}} h(x+d)$ converges absolutely and uniformly on compact sets and is infinitely differentiable as a function of x, the *Poisson summation formula* is

$$\sum_{n \in \mathbb{Z}} h(x+n) = \sum_{n \in \mathbb{Z}} \hat{h}(n) e^{2\pi i n x}.$$

The idea here is that the left side is the periodicization of h, and then the right side is the Fourier series of the left side, because the nth Fourier coefficient of the periodicization of h is the nth Fourier transform of h itself. When $x = 0$ the Poisson summation formula specializes to

$$\sum_{n \in \mathbb{Z}} h(n) = \sum_{n \in \mathbb{Z}} \hat{h}(n).$$

And especially, if $h(x)$ if the Gaussian $f(xt^{1/2})$ then Poisson summation with $x = 0$ shows that

$$\sum_{n \in \mathbb{Z}} f(nt^{1/2}) = t^{-1/2} \sum_{n \in \mathbb{Z}} f(nt^{-1/2}),$$

which is to say,

$$\vartheta(i/t) = t^{1/2} \vartheta(it), \quad t > 0. \tag{1}$$

Riemann zeta as the Mellin transform of theta. With these preliminaries in hand, the properties of the Riemann zeta function are established by examining the *Mellin transform* of (essentially) the theta function. In general, the Mellin transform of a function $f : \mathbb{R}^+ \longrightarrow \mathbb{C}$ is the integral

$$g(s) = \int_{t=0}^{\infty} f(t) t^{s/2} \frac{dt}{t}$$

for s-values such that the integral converges absolutely. For example, the Mellin transform of e^{-t} is $\Gamma(s/2)$. Also, the Mellin transform of the function

$$\sum_{n=1}^{\infty} e^{-\pi n^2 t} = \tfrac{1}{2}(\vartheta(it) - 1), \quad t > 0$$

is

$$g(s) = \int_{t=0}^{\infty} \sum_{n=1}^{\infty} e^{-\pi n^2 t} t^{s/2} \frac{dt}{t}.$$

Since $\vartheta(it)$ converges to 1 as $t \to \infty$, the transformation law (1) shows that as $t \to 0$, $\vartheta(it)$ grows at the same rate as $t^{-1/2}$, and therefore the integral $g(s)$ converges at its left endpoint if $\mathrm{Re}\,(s) > 1$. And since the convergence of $\vartheta(it)$ to 1 as $t \to \infty$ is rapid, the integral converges at its right end for all values of s. Rapid convergence lets the sum pass through the integral in the previous display to yield, after a change of variable,

$$g(s) = \sum_{n=1}^{\infty} (\pi n^2)^{-s/2} \int_{t=0}^{\infty} e^{-t} t^{s/2} \frac{dt}{t} = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}\,(s) > 1.$$

Thus, when $\mathrm{Re}\,(s) > 1$, the integral $g(s)$ is the function $\xi(s)$ mentioned at the beginning of this writeup. On the other hand, recall that the function whose Mellin transform we took is essentially the theta function,

$$\sum_{n=1}^{\infty} e^{-\pi n^2 t} = \tfrac{1}{2}(\vartheta(it) - 1), \quad t > 0.$$

So this paragraph has in fact shown that the modified zeta function

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}\,(s) > 1$$

has an integral representation as the Mellin transform of the theta function,

$$\xi(s) = \frac{1}{2} \int_{t=0}^{\infty} (\vartheta(it) - 1) t^{s/2} \frac{dt}{t}, \quad \mathrm{Re}\,(s) > 1.$$

Thinking in these terms, the factor $\pi^{-s/2}\Gamma(s/2)$ is intrinsically associated to $\zeta(s)$, making $\xi(s)$ the natural function to consider. Modern adelic considerations make the factor even more natural, but those ideas are beyond our current scope.

Meromorphic continuation. The integral representation of $\xi(s)$ provides its meromorphic continuation and functional equation. Compute part of the integral by splitting off a term, replacing t by $1/t$, using the transformation law (1) for $\vartheta(it)$, and splitting off another term to resymmetrize,

$$\begin{aligned}
\frac{1}{2} \int_{t=0}^{1} (\vartheta(it) - 1) t^{s/2} \frac{dt}{t} &= \frac{1}{2} \int_{t=0}^{1} \vartheta(it) t^{s/2} \frac{dt}{t} - \frac{1}{s} \\
&= \frac{1}{2} \int_{t=1}^{\infty} \vartheta(i/t) t^{-s/2} \frac{dt}{t} - \frac{1}{s} \\
&= \frac{1}{2} \int_{t=1}^{\infty} \vartheta(it) t^{(1-s)/2} \frac{dt}{t} - \frac{1}{s} \\
&= \frac{1}{2} \int_{t=1}^{\infty} (\vartheta(it) - 1) t^{(1-s)/2} \frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}.
\end{aligned}$$

Combine this with the remainder of the integral representation of $\xi(s)$ to get

$$\xi(s) = \frac{1}{2}\int_{t=1}^{\infty}(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}, \quad \mathrm{Re}\,(s) > 1.$$

But since the integral in the last display now has as its left endpoint of integration $t = 1$ rather than $t = 0$, it is entire in s, making the right side meromorphic everywhere in the s-plane with its only poles being simple poles at $s = 0$ and $s = 1$. That is, the new description of ξ is no longer constrained to the domain $\{\mathrm{Re}\,(s) > 1\}$,

$$\boxed{\xi(s) = \frac{1}{2}\int_{t=1}^{\infty}(\vartheta(it)-1)(t^{s/2}+t^{(1-s)/2})\frac{dt}{t} - \frac{1}{s} - \frac{1}{1-s}, \quad s \in \mathbb{C}.}$$

The new description extends ξ to a meromorphic function on all of $\mathbb{C}$. The definition of the extended function no longer makes reference to $\zeta(s)$ as a sum.

Functional equation. Finally, the right side of the boxed display is clearly invariant under the substitution $s \longmapsto 1-s$. That is, the meromorphic continuation of $\xi(s)$ to the full s-plane satisfies the functional equation

$$\boxed{\xi(1-s) = \xi(s), \quad s \in \mathbb{C}.}$$