COMPLEX TORI AS ELLIPTIC CURVES

This writeup shows how complex tori $\mathbb{C}/\Lambda$ can also be viewed as cubic curves. These cubic curves are called *elliptic* despite not being ellipses, due to a connection between them and the arc length of an actual ellipse. The presentation here is terse, so you may want to consult a relevant complex analysis text.

1. The Weierstrass $\wp$-function and its Derivative

The meromorphic functions on a complex torus are what relate it to a cubic curve. Given a lattice Λ, the meromorphic functions $f : \mathbb{C}/\Lambda \longrightarrow \widehat{\mathbb{C}}$ on the torus are naturally identified with the Λ-periodic meromorphic functions $f : \mathbb{C} \longrightarrow \widehat{\mathbb{C}}$ on the plane. Exercise 1 derives some basic properties of these functions in general. The most important specific example is the *Weierstrass $\wp$-function*

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda}{}' \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right), \quad z \in \mathbb{C},\ z \notin \Lambda.$$

(The primed summation means to omit $\lambda = 0$.) Subtracting $1/\lambda^2$ from $1/(z-\lambda)^2$ makes the summand roughly z/λ^3, cf. the sketched proof of Proposition 1.1 to follow, so the sum converges absolutely and uniformly on compact subsets of $\mathbb{C}$ away from Λ. Correcting the summand this way prevents the terms of the sum from being permuted when z is translated by a lattice element, so $\wp$ doesn't obviously have periods Λ. But its derivative

$$\wp'(z) = -2\sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^3}$$

clearly does have periods Λ, and combining this with the fact that $\wp$ is an even function quickly shows that in fact $\wp$ has periods Λ as well (Exercise 2). It turns out that $\wp$ and $\wp'$ are the only basic examples we need, because the field of meromorphic functions on $\mathbb{C}/\Lambda$ is $\mathbb{C}(\wp, \wp')$, the rational expressions in these two functions. Because the Weierstrass $\wp$-function depends on the lattice Λ as well as the variable z we will sometimes write $\wp_\Lambda(z)$ and $\wp'_\Lambda(z)$; in particular for lattices Λ_τ we will write $\wp_\tau(z)$ and $\wp'_\tau(z)$.

Eisenstein series are functions of lattices,

$$G_k(\Lambda) = \sum_{\lambda\in\Lambda}{}' \frac{1}{\lambda^k}, \quad k > 2 \text{ even}.$$

They satisfy the *homogeneity condition* $G_k(m\Lambda) = m^{-k} G_k(\Lambda)$ for all $m \in \mathbb{C}^\times$. Part (a) of the next result shows that Eisenstein series appear in the Laurent expansion of the Weierstrass $\wp$-function for Λ. Part (b) relates the functions $\wp(z)$ and $\wp'(z)$ in a cubic equation whose coefficients are also Eisenstein series.

Proposition 1.1. *Let $\wp$ be the Weierstrass function with respect to a lattice Λ. Then*

(a) *The Laurent expansion of* $\wp$ *is*

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} (n+1)G_{n+2}(\Lambda)z^n$$

for all z *such that* $0 < |z| < \inf\{|\lambda| : \lambda \in \Lambda \setminus \{0\}\}$.

(b) *The functions* $\wp$ *and* $\wp'$ *satisfy the relation*

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda)$$

where $g_2(\Lambda) = 60G_4(\Lambda)$ *and* $g_3(\Lambda) = 140G_6(\Lambda)$.

(c) *Let* $\Lambda = \lambda_1\mathbb{Z} \oplus \lambda_2\mathbb{Z}$ *and let* $\lambda_3 = \lambda_1 + \lambda_2$. *Then the cubic equation satisfied by* $\wp$ *and* $\wp'$, $y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$, *is*

$$y^2 = 4(x-e_1)(x-e_2)(x-e_3), \qquad e_i = \wp(\lambda_i/2) \text{ for } i = 1,2,3.$$

This equation is nonsingular, meaning its right side has distinct roots.

Proof. (Sketch.) For (a), if $|z| < |\lambda|$ then

$$\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}\left(\frac{1}{(1-z/\lambda)^2} - 1\right)$$

and the geometric series squares to $\sum_{n=0}^{\infty}(n+1)z^n/\lambda^n$, making the summand $\sum_{n=1}^{\infty}(n+1)z^n/\lambda^{n+2}$. The double sum $\sum_{\lambda}' \sum_{n=1}^{\infty}(n+1)z^n/\lambda^{n+2}$ can be rearranged to $\sum_{n=1}^{\infty}\sum_{\lambda}'(n+1)z^n/\lambda^{n+2} = \sum_{n=1}^{\infty}(n+1)G_{k+2}(\Lambda)z^n$, and $G_{k+2}(\Lambda) = 0$ when n is odd.

For (b), one uses part (a) to show that the nonpositive terms of the Laurent series of both sides are equal. Specifically, because

$$\wp(z) = \frac{1}{z^2} + 3G_4(\Lambda)z^2 + 5G_6(\Lambda)z^4 + \mathcal{O}(z^6)$$

(where "$\mathcal{O}$" means "a quantity on the order of") and

$$\wp'(z) = -\frac{2}{z^3} + 6G_4(\Lambda)z + 20G_6(\Lambda)z^3 + \mathcal{O}(z^5),$$

a little algebra shows that both $(\wp'(z))^2$ and $4(\wp(z))^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda)$ work out to $4/z^6 - 24G_4(\Lambda)/z^2 - 80G_6(\Lambda) + \mathcal{O}(z^2)$. So their difference is holomorphic and Λ-periodic, therefore bounded, therefore constant, therefore zero because it is $\mathcal{O}(z^2)$ as $z \to 0$.

For (c), because $\wp'$ is odd, it has zeros at the order 2 points of $\mathbb{C}/\Lambda$: if $z \equiv -z \pmod{\Lambda}$ then $\wp'(z) = \wp'(-z) = -\wp'(z)$ and thus $\wp'(z) = 0$. Letting $\Lambda = \lambda_1\mathbb{Z}\oplus\lambda_2\mathbb{Z}$, the order 2 points are $z_i = \lambda_i/2$ with $\wp'(z_i) = 0$ for $i = 1,2,3$. The relation between $\wp$ and $\wp'$ from (b) shows that the corresponding values $x_i = \wp(z_i)$ for $i = 1,2,3$ are roots of the cubic polynomial $p_\Lambda(x) = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$, so it factors as claimed. Each x_i is a double value of $\wp$ because $\wp'(z_i) = 0$, and because $\wp$ has degree 2, meaning it takes each value twice counting multiplicity (see Exercise 1(b)), this makes the three x_i distinct. That is, the cubic polynomial p_Λ has distinct roots. $\square$

2. Complex Tori and Elliptic Curves

Part (b) of the proposition shows that the map $z \mapsto (\wp_\Lambda(z), \wp'_\Lambda(z))$ takes non-lattice points of $\mathbb{C}$ to points $(x, y) \in \mathbb{C}^2$ satisfying the nonsingular cubic equation of part (c), $y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda)$. The map bijects because generally a value $x \in \mathbb{C}$ is taken by $\wp_\Lambda$ twice on $\mathbb{C}/\Lambda$, that is, $x = \wp_\Lambda(\pm z + \Lambda)$, and then the two y-values satisfying the cubic equation are $\wp'(\pm z + \Lambda) = \pm\wp'(z + \Lambda)$. The exceptional x-values where $y = 0$ occur at the order-2 points of $\mathbb{C}/\Lambda$, so they are taken once by $\wp_\Lambda$ as necessary. The map extends to all $z \in \mathbb{C}$ by mapping lattice points to a suitably defined point at infinity. In sum, for every lattice the associated Weierstrass $\wp$-function and its derivative give a bijection

$$(\wp, \wp') : \text{complex torus} \longrightarrow \text{elliptic curve}.$$

For example, the value $g_3(i) = 0$ shows that the complex torus $\mathbb{C}/\Lambda_i$ bijects to the elliptic curve with equation $y^2 = 4x^3 - g_2(i)x$. Similarly the complex torus $\mathbb{C}/\Lambda_{\zeta_3}$ (where again $\zeta_3 = e^{2\pi i/3}$) bijects to the elliptic curve with equation $y^2 = 4x^3 - g_3(\zeta_3)$. See Exercise 3 for some values of the functions $\wp$ and $\wp'$ in connection with these two lattices.

The map $(\wp, \wp')$ transfers the group law from the complex torus to the elliptic curve. To understand addition on the curve, let $z_1 + \Lambda$ and $z_2 + \Lambda$ be nonzero points of the torus. The image points $(\wp(z_1), \wp'(z_1))$ and $(\wp(z_2), \wp'(z_2))$ on the curve determine a secant or tangent line of the curve in $\mathbb{C}^2$, $ax + by + c = 0$. Consider the function

$$f(z) = a\wp(z) + b\wp'(z) + c.$$

This is meromorphic on $\mathbb{C}/\Lambda$. When $b \neq 0$ it has a triple pole at $0 + \Lambda$ and zeros at $z_1 + \Lambda$ and $z_2 + \Lambda$, and Exercise 1(c) shows that its third zero is at the point $z_3 + \Lambda$ such that $z_1 + z_2 + z_3 + \Lambda = 0 + \Lambda$ in $\mathbb{C}/\Lambda$. When $b = 0$, f has a double pole at $0 + \Lambda$ and zeros at $z_1 + \Lambda$ and $z_2 + \Lambda$, and Exercise 1(c) shows that $z_1 + z_2 + \Lambda = 0 + \Lambda$ in $\mathbb{C}/\Lambda$. In this case let $z_3 = 0 + \Lambda$ so that again $z_1 + z_2 + z_3 + \Lambda = 0 + \Lambda$, and because the line is vertical view it as containing the infinite point $(\wp(0), \wp'(0))$ whose second coordinate arises from a pole of higher order than the first. Thus for any value of b the elliptic curve points on the line $ax + by + c = 0$ are the points $(x_i, y_i) = (\wp(z_i), \wp'(z_i))$ for $i = 1, 2, 3$. Because $z_1 + z_2 + z_3 + \Lambda = 0 + \Lambda$ on the torus in all cases, the resulting group law on the curve is:

Collinear triples sum to zero.

Recall that a holomorphic isomorphism of complex tori takes the form

$$z + \Lambda \mapsto mz + \Lambda' \quad \text{where } \Lambda' = m\Lambda.$$

Because $\wp_{m\Lambda}(mz) = m^{-2}\wp_\Lambda(z)$ and $\wp'_{m\Lambda}(mz) = m^{-3}\wp'_\Lambda(z)$, the corresponding isomorphism of elliptic curves is

$$(x, y) \mapsto (m^{-2}x, m^{-3}y),$$

or equivalently the substitution

$$(x, y) = (m^2x', m^3y'),$$

changing the cubic equation $y^2 = 4x^3 - g_2x - g_3$ associated to Λ to the equation $y^2 = 4x^3 - m^{-4}g_2x - m^{-6}g_3$ associated to Λ'. Suitable choices of m (Exercise 3 again) normalize the elliptic curves associated to $\mathbb{C}/m\Lambda_i$ and $\mathbb{C}/m\Lambda_{\zeta_3}$ to have equations

$$y^2 = 4x(x-1)(x+1), \qquad y^2 = 4(x-1)(x-\zeta_3)(x-\zeta_3^2).$$

The appearance of Eisenstein series as coefficients of a nonsingular curve lets us prove as a corollary to Proposition 1.1 that the discriminant function has no zeros in the upper half plane.

Corollary 2.1. *The function Δ is nonvanishing on $\mathcal{H}$. That is, $\Delta(\tau) \neq 0$ for all $\tau \in \mathcal{H}$.*

Proof. For any $\tau \in \mathcal{H}$, specialize the lattice Λ in the proposition to Λ_τ. By part (c) of the proposition, the cubic polynomial $p_\tau(x) = 4x^3 - g_2(\tau)x - g_3(\tau)$ has distinct roots. Exercise 4 shows that $\Delta(\tau)$ is the discriminant of p_τ up to constant multiple (hence its name), so $\Delta(\tau) \neq 0$. $\square$

Not only does every complex torus $\mathbb{C}/\Lambda$ lead via the Weierstrass $\wp$-function to an elliptic curve

$$y^2 = 4x^3 - a_2x - a_3, \quad a_2^3 - 27a_3^2 \neq 0 \tag{1}$$

with $a_2 = g_2(\Lambda)$ and $a_3 = g_3(\Lambda)$, but the converse holds as well.

Proposition 2.2. *Given an elliptic curve* (1)*, there exists a lattice Λ such that $a_2 = g_2(\Lambda)$ and $a_3 = g_3(\Lambda)$.*

Proof. The case $a_2 = 0$ and the case $a_3 = 0$ are Exercise 5. For the case $a_2 \neq 0$ and $a_3 \neq 0$, because $j : \mathcal{H} \longrightarrow \mathbb{C}$ surjects there exists $\tau \in \mathcal{H}$ such that $j(\tau) = 1728a_2^3/(a_2^3 - 27a_3^2)$. This gives

$$\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2} = \frac{a_2^3}{a_2^3 - 27a_3^2},$$

or, after taking reciprocals and doing a little algebra,

$$\frac{a_2^3}{g_2(\tau)^3} = \frac{a_3^2}{g_3(\tau)^2}. \tag{2}$$

For any nonzero complex number λ_2, let $\lambda_1 = \tau\lambda_2$ and $\Lambda = \lambda_1\mathbb{Z} \oplus \lambda_2\mathbb{Z}$. Then

$$g_2(\Lambda) = \lambda_2^{-4}g_2(\tau) \quad \text{and} \quad g_3(\Lambda) = \lambda_2^{-6}g_3(\tau).$$

Thus we are done if we can choose λ_2 such that

$$\lambda_2^{-4} = a_2/g_2(\tau) \quad \text{and} \quad \lambda_2^{-6} = a_3/g_3(\tau).$$

Choose λ_2 to satisfy the first condition, so that $\lambda_2^{-12} = a_2^3/g_2(\tau)^3$. Then by (2) $\lambda_2^{-6} = \pm a_3/g_3(\tau)$, and replacing λ_2 by $i\lambda_2$ if necessary completes the proof. $\square$

It follows that any map of elliptic curves $(x, y) \mapsto (m^{-2}x, m^{-3}y)$, changing a cubic equation $y^2 = 4x^3 - a_2x - a_3$ to the equation $y^2 = 4x^3 - m^{-4}a_2x - m^{-6}a_3$, comes from the holomorphic isomorphism of complex tori $z + \Lambda \mapsto mz + \Lambda'$ where $a_2 = g_2(\Lambda)$, $a_3 = g_3(\Lambda)$, and $\Lambda' = m\Lambda$. This makes the map of elliptic curves an isomorphism as well.

Thus complex tori (Riemann surfaces, complex analytic objects) and elliptic curves (solution sets of cubic polynomials, algebraic objects) are interchangeable. With the connection between them in hand, let the term *complex elliptic curve* be a synonym for *complex torus* and call meromorphic functions with periods Λ *elliptic functions with respect to* Λ.

Exercises.

(1) Let $E = \mathbb{C}/\Lambda$ be a complex elliptic curve where $\Lambda = \lambda_1\mathbb{Z} \oplus \lambda_2\mathbb{Z}$, and let f be a nonconstant elliptic function with respect to Λ, viewed either as a meromorphic function on $\mathbb{C}$ with periods Λ or as a meromorphic function on E. Let $P = \{x_1\lambda_1 + x_2\lambda_2 : x_1, x_2 \in [0,1]\}$ be the parallelogram representing E when its opposing boundary edges are suitably identified, and let ∂P be the counterclockwise boundary of P. Because f has only finitely many zeros and poles in E, some translation $t + \partial P$ misses them all. This exercise establishes some necessary properties of f. Showing that these properties are sufficient for an appropriate f to exist requires more work.

(a) Compute that $1/(2\pi i)\int_{t+\partial P} f(z)dz = 0$. It follows by the Residue Theorem that the sum of the residues of f on E is 0. In particular there is no meromorphic function on E with one simple pole and so the Weierstrass $\wp$-function, with its double pole at Λ, is the simplest nonconstant elliptic function with respect to Λ.

(b) Compute that $1/(2\pi i)\int_{t+\partial P} f'(z)dz/f(z) = 0$. It follows by the Argument Principle that f has as many zeros as poles, counting multiplicity. Replacing f by $f - w$ for any $w \in \mathbb{C}$ shows that f takes every value the same number of times, counting multiplicity. In particular the Weierstrass $\wp$-function on E takes every value twice.

(c) Compute that $1/(2\pi i)\int_{t+\partial P} zf'(z)dz/f(z) \in \Lambda$. Show that this integral is also $\sum_{x\in E}\nu_x(f)x$ where $\nu_x(f)$ is the order of f at x, meaning that $f(z) = (z-x)^{\nu_x(f)}g(z)$ with $g(x) \neq 0$. Note that $\nu_x(f) = 0$ except at zeros and poles of f, so the sum is finite. Thus parts (b) and (c) combine to show that for any nonconstant meromorphic function f on E,

$$\sum_{x\in E}\nu_x(f) = 0 \text{ in } \mathbb{Z} \quad \text{and} \quad \sum_{x\in E}\nu_x(f)x = 0 \text{ in } E.$$

(2) Let $\Lambda = \lambda_1\mathbb{Z} \oplus \lambda_2\mathbb{Z}$ be a lattice and let $\wp$ be its associated Weierstrass $\wp$-function.

(a) Show that $\wp$ is even and that $\wp'$ is Λ-periodic.

(b) For $i = 1, 2$ show that the function $\wp(z + \lambda_i) - \wp(z)$ is some constant c_i by taking its derivative. Substitute $z = -\lambda_i/2$ to show that $c_i = 0$. Conclude that $\wp$ is Λ-periodic.

(3) Let $\Lambda = \Lambda_i$. The derivative $\wp'$ of the corresponding Weierstrass function has a triple pole at 0 and simple zeros at $1/2$, $i/2$, $(1+i)/2$. Because $i\Lambda = \Lambda$ it follows that

$$\wp(iz) = \wp_{i\Lambda}(iz) = i^{-2}\wp(z) = -\wp(z).$$

Show that in particular $\wp((1+i)/2) = 0$. This is a double zero of $\wp$ because also $\wp'((1+i)/2) = 0$, making it the only zero of $\wp$ as a function on $\mathbb{C}/\Lambda$. Because $\overline{\Lambda} = \Lambda$ (complex conjugation) it also follows that $\wp(\overline{z}) = \overline{\wp(z)}$, so that $\wp(1/2)$ is real, as is $\wp(i/2) = -\wp(1/2)$. Compute some dominant terms of $\wp(1/2)$ and $\wp(i/2)$ to show that $\wp(1/2)$ is the positive value. For what m does the complex torus $\mathbb{C}/m\Lambda$ correspond to the elliptic curve with equation $y^2 = 4x(x-1)(x+1)$?

Reason similarly with $\Lambda = \Lambda_{\zeta_3}$ to find the zeros of the corresponding Weierstrass function $\wp$ and to show that $\wp(1/2)$ is real. For what m does the complex torus $\mathbb{C}/m\Lambda$ correspond to the elliptic curve with equation $y^2 = 4(x-1)(x-\zeta_3)(x-\zeta_3^2)$?

(4) For $\tau \in \mathcal{H}$ let $p_\tau(x) = 4x^3 - g_2(\tau)x - g_3(\tau)$. Show that the discriminant of p_τ equals $\Delta(\tau)$ up to constant multiple, where Δ is the cusp form of weight 12.

(5) Show that when $a_2 = 0$ in Proposition 2.2 the desired lattice is $\Lambda = m\Lambda_{\zeta_3}$ for a suitably chosen m. Prove the case $a_3 = 0$ in Proposition 2.2 similarly.