STEREOGRAPHIC PROJECTION AND THE MERCATOR MAP

1. Stereographic Projection

Let

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

be the unit sphere, and let $\mathbf{n}$ denote the north pole $(0, 0, 1)$. Identify the complex plane $\mathbb{C}$ with the (x, y)-plane in $\mathbb{R}^3$.

The *stereographic projection* map,

$$\pi : S^2 - \mathbf{n} \longrightarrow \mathbb{C},$$

is described as follows: place a light source at the north pole $\mathbf{n}$. For any point $(x, y, z) \in S^2 - \mathbf{n}$, consider a light ray emanating downward from $\mathbf{n}$ to pass through the sphere at (x, y, z). The ray also hits the plane, and the point where it hits is designated $\pi(x, y, z)$. The formula works out to

$$\pi(x, y, z) = \frac{x + iy}{1 - z}.$$

2. The Mercator Map

A *loxodrome* is the path of a ship that sails in a fixed compass-direction. (See Figure 1.)

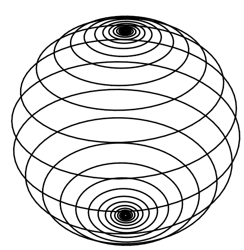

Figure 1. Loxodrome

The parametrization of the unit sphere in terms of its longitude θ (east-west angle) and latitude φ (north-south angle) is

$$(\theta, \varphi) \longmapsto (\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi) = (x, y, z).$$

Here $-\pi < \theta \le \pi$ and $-\pi/2 \le \varphi \le \pi/2$. Place the unit sphere inside an infinite vertical cylinder with coordinates (r, θ), where the r-axis points north and θ is again the longitude. The *Mercator map* from the parameter space of the sphere (excluding $\varphi = \pm\pi/2$) to the cylinder is by definition the parametrization followed by the map that takes the equator to itself, preserves the longitude coordinate θ, and takes loxodromes on the sphere to lines in (r, θ)-space.

This map is *not* radial projection from the center of the sphere out to the cylinder. Instead, non-obviously, the map is given by

$$r(\varphi) = \log(\tan\varphi + \sec\varphi). \tag{1}$$

(The logarithm is natural, of course.) To see this, first consider latitudinal motion through distance Δs. (See figure 2, depicting the side view of a vertical cross section of the sphere inside the cylinder, through the center of the sphere, at any longitude.) The figure shows that

$$\Delta s = \Delta\varphi,$$

and so the limiting rate of change of the Mercator map coordinate r with respect to latitudinal distance on the sphere is

$$\frac{dr}{ds} = \lim_{\Delta s \to 0} \frac{\Delta r}{\Delta s} = \lim_{\Delta\varphi \to 0} \frac{\Delta r}{\Delta\varphi} = r'(\varphi). \tag{2}$$

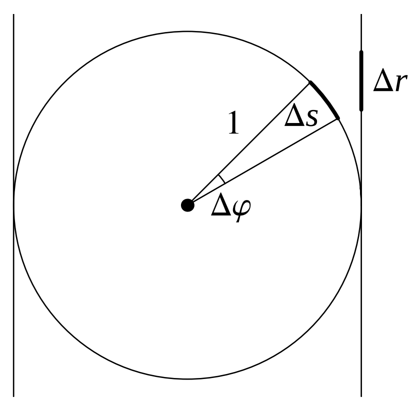

FIGURE 2. Latitudinal motion

Also consider longitudinal motion through distance Δs. (See figure 3, depicting the top view of the horizontal cross section of the sphere inside the cylinder at latitude φ.) The figure shows that

$$\Delta s = \Delta\theta \cos\varphi,$$

so that

$$\frac{\Delta\theta}{\Delta s} = \sec\varphi,$$

and therefore the limiting rate of change of the Mercator map coordinate θ with respect to longitudinal distance on the sphere is

$$\frac{d\theta}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\theta}{\Delta s} = \sec\varphi. \tag{3}$$

Since dr/ds measures the response of one Mercator map coordinate to longitudinal motion, and $d\theta/ds$ measures the response of the other Mercator map coordinate

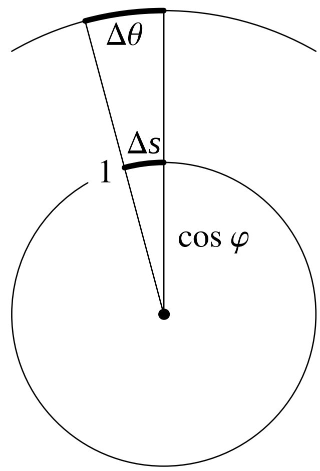

FIGURE 3. Longitudinal motion at latitude φ

to latitudinal motion, *the defining characteristic of the Mercator map is that they are equal.* By (2) and (3) this condition is

$$r'(\varphi) = \sec\varphi.$$

The antiderivative of sec is $\log(\tan + \sec)$, giving the claimed formula (1) for r.

3. The Relation Between the Two Maps

Identify the Mercator image point

$$(r, \theta) = (\log(\tan\varphi + \sec\varphi), \theta)$$

with the point $r + i\theta \in \mathbb{C}$. The image of this point under the exponential map is

$$e^{r+i\theta} = e^r e^{i\theta} = (\tan\varphi + \sec\varphi)e^{i\theta}.$$

On the other hand, parametrize the sphere and then project stereographically to get

$$\begin{aligned}(\theta, \varphi) &\longmapsto (\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi)\\ &\longmapsto \frac{\cos\varphi}{1-\sin\varphi}(\cos\theta + i\sin\theta) = (\tan\varphi + \sec\varphi)e^{i\theta}.\end{aligned}$$

Since the two results are the same, the following diagram commutes, where now the prepended parametrization of the sphere has been dropped from the Mercator map, making its domain the sphere other than the north and south poles:

$$\begin{tikzcd} & S^2 - \{\mathbf{n}, \mathbf{s}\} \arrow[dl, "\text{Mercator}"'] \arrow[dr, "\text{Stereographic}"] & \\ \mathbb{C} \arrow[rr, "\exp"] & & \mathbb{C} \end{tikzcd}$$