# LEGENDRE DUPLICATION FORMULA

The beta function is

$$B(a,b) = \int_{x=0}^{1} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x, \quad a>0,\ b>0.$$

Compute, with $x = \frac{1+y}{2}$ at the second step to follow and then with $x = y^2$ so that $\mathrm{d}x = 2y\,\mathrm{d}y$ and thus $\mathrm{d}y = \frac{1}{2x^{1/2}}\,\mathrm{d}x = \frac{1}{2}x^{1/2-1}\,\mathrm{d}x$ at the fifth step, that for $b > 0$,

$$\begin{aligned} B(b,b) &= \int_{x=0}^{1} \big(x(1-x)\big)^{b-1}\,\mathrm{d}x \\ &= \frac{1}{2}\int_{y=-1}^{1} \left(\frac{1+y}{2}\cdot\frac{1-y}{2}\right)^{b-1} dy \\ &= 2^{1-2b}\int_{y=-1}^{1} (1-y^2)^{b-1}\,dy \\ &= 2^{2-2b}\int_{y=0}^{1} (1-y^2)^{b-1}\,dy \\ &= 2^{1-2b}\int_{x=0}^{1} x^{1/2-1}(1-x)^{b-1}\,dx \\ &= 2^{1-2b}B(\tfrac{1}{2},b). \end{aligned}$$

Repeating,

$$B(b,b) = 2^{1-2b}B(\tfrac{1}{2},b), \quad b>0. \tag{1}$$

Also, we will show below that

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b), \quad a>0,\ b>0. \tag{2}$$

It follows that for all $s > 0$,

$$\begin{aligned} \Gamma(\tfrac{s}{2})^2 &= \Gamma(s)B(\tfrac{s}{2},\tfrac{s}{2}) && \text{by (2) with } a=b=\tfrac{s}{2} \\ &= \Gamma(s)2^{1-s}B(\tfrac{1}{2},\tfrac{s}{2}) && \text{by (1) with } b=\tfrac{s}{2} \\ &= \Gamma(s)2^{1-s}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})} && \text{by (2) with } a=\tfrac{1}{2},\ b=\tfrac{s}{2}. \end{aligned}$$

Because $\Gamma(\frac{1}{2}) = \pi^{1/2}$, this gives Legendre's formula $\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = 2^{1-s}\pi^{1/2}\Gamma(s)$ for $s > 0$. And because $\Gamma(s)/\big(\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})\big)$ is entire, this relation extends meromorphically to the full complex plane,

$$\boxed{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\pi^{1/2}\Gamma(s), \quad s\in\mathbb{C}.}$$

To complete the argument, we establish (2). Compute for $a, b > 0$, using Fubini's theorem and the Haar measure property $\mathrm{d}(cz)/(cz) = \mathrm{d}z/z$ freely, that

$$\begin{aligned}
\Gamma(a)\Gamma(b) &= \int_{t>0} e^{-t} t^a \frac{\mathrm{d}t}{t} \int_{u>0} e^{-u} u^b \frac{\mathrm{d}u}{u} \\
&= \int_{t>0} \int_{u>0} e^{-t-u} t^a u^b \frac{\mathrm{d}u}{u} \frac{\mathrm{d}t}{t} \\
&= \int_{t>0} \int_{u>0} e^{-t-tu} t^a (tu)^b \frac{\mathrm{d}u}{u} \frac{\mathrm{d}t}{t} \\
&= \int_{u>0} \int_{t>0} e^{-(1+u)t} t^{a+b} \frac{\mathrm{d}t}{t} u^b \frac{\mathrm{d}u}{u} \\
&= \int_{u>0} \int_{t>0} e^{-t} \left(\frac{t}{1+u}\right)^{a+b} \frac{\mathrm{d}t}{t} u^b \frac{\mathrm{d}u}{u} \\
&= \int_{t>0} e^{-t} t^{a+b} \frac{\mathrm{d}t}{t} \int_{u>0} \left(\frac{1}{1+u}\right)^{a+b} u^b \frac{\mathrm{d}u}{u} \\
&= \Gamma(a+b) \int_{u>0} \left(\frac{1}{1+u}\right)^{a+1} \left(\frac{u}{1+u}\right)^{b-1} \mathrm{d}u.
\end{aligned}$$

Let $x = 1/(1+u)$, so that $x$ goes from 1 to 0 and $\mathrm{d}u = \mathrm{d}(1/x - 1) = -\mathrm{d}x/x^2$, to get the desired result,

$$\Gamma(a)\Gamma(b) = \Gamma(a+b) \int_{x=0}^{1} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x = \Gamma(a+b)B(a,b).$$

As an end note, we observe that the methods here again establish the symmetry formula for the gamma function. Specifically, for $0 < s < 1$, the long computation just shown also gives, with $a = s$ and $b = 1 - s$, now denoting the variable of integration $x$ rather than $u$,

$$\Gamma(s)\Gamma(1-s) = \Gamma(1) \int_{x>0} \left(\frac{1}{1+x}\right)^{s+1} \left(\frac{x}{1+x}\right)^{-s} \mathrm{d}x = \int_{x>0} \frac{x^{-s}}{1+x}\,\mathrm{d}x.$$

We have evaluated this last integral by contour integration and then noted that the resulting identity extends to all $s$,

$$\boxed{\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad s \in \mathbb{C}.}$$