MATH 311: COMPLEX ANALYSIS — PREVIEW LECTURE

1. Introduction

Complex analysis: not very complicated, not much analysis.

The analysis is what's called "soft" analysis – some integrals and derivatives, but very few epsilons and grungy estimates once Cauchy's Theorem is proved.

Real Analysis (Math 321) is not a prerequisite for this course, but students who haven't had it will sometimes have to accept results and methods on faith or do extra background work.

The subject: differentiable functions

$$f : \Omega \longrightarrow \mathbb{C},$$

where $\Omega \subset \mathbb{C}$ is a *region*, meaning a connected open set. "Connected" doesn't necessarily mean "simply connected," i.e., the region Ω can have holes.

Complex-differentiable functions are subject to a tremendous amount of structure.

For example:

Theorem 1.1 (Uniqueness Theorem)**.** *Suppose that the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is differentiable, and*

$$f(1) = 0, \quad f(1/2) = 0, \quad f(1/3) = 0, \quad f(1/4) = 0, \quad \dots$$

Then $f = 0$.

Compare against $f : \mathbb{R} \longrightarrow \mathbb{R}$ where

$$f(x) = \begin{cases} x^2 \sin(\pi/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Or:

Theorem 1.2 (Liouville's Theorem)**.** *Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is differentiable and bounded.*

Then f is constant.

By contrast, there are many differentiable, bounded, nonconstant functions from $\mathbb{R}$ to $\mathbb{R}$.

Or:

Theorem 1.3 (Picard's Theorem)**.** *Suppose $f : \mathbb{C} \longrightarrow \mathbb{C}$ is differentiable and misses two points.*

Then f is constant.

So apparently there aren't a lot of differentiable functions.

And yet there are enough to do interesting things and solve interesting problems.

2. Some applications

1. **Work integrals and sums in $\mathbb{R}$**:

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx,$$

or

$$\int_0^{2\pi} R(\sin\theta,\cos\theta)\,d\theta \qquad \text{where } R \text{ is a rational function,}$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^2}. \qquad \text{(This converges by the integral test, but to what?)}$$

Note that these problems are set in the reals. But we solve them using complex numbers.

2. **Dirichlet's Problem on the disk**: Let

$$D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

be the unit disk, with boundary

$$\partial D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Given a piecewise continuous function

$$\varphi : \partial D \longrightarrow \mathbb{R},$$

find all functions

$$f : D \cup \partial D \longrightarrow \mathbb{R}$$

such that

- $f = \varphi$ on ∂D, i.e., f extends φ,
- f is differentiable on D and continuous on $D \cup \partial D$ except at the original discontinuities of φ,
- $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ on D, where $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the *Laplacian* operator.

Again this is a problem in the reals.

3. **The Riemann Mapping Problem**: Given Ω a simply connected planar region not equal to all of $\mathbb{R}^2$, show that there is a conformal map

$$\Omega \xrightarrow{\sim} D.$$

This is true and astonishing. "Conformal" means angle-preserving, so, for example, the interior of a stop-sign is conformally equivalent to the disk even though the stop sign has corners.

"Dirichlet's Comb" is an even more surprising example.

4. **Models for hyperbolic geometry**: Define notions of point, line, congruence, etc. in the unit disk D such that the first four postulates of Euclidean geometry hold, but the fifth postulate (the *parallel postulate*) fails.

5. **Number Theory**: Surprisingly, many questions about the integers $\mathbb{Z}$ can be addressed through complex analytic techniques. For example,

- The *Prime Number Theorem* states that
$$\frac{\pi(n)}{n/\log n} \to 1 \quad \text{as } n \to \infty,$$
where $\pi(n)$ is the number of primes less than n and log is the natural logarithm. This can be proved with Riemann's zeta-function.
- *Dirichlet's Theorem on Arithmetic Progressions* says that given integers a and N satisfying $1 \le a < N$ and $\gcd(a, N) = 1$, there exist infinitely many primes p such that
$$p \equiv a \pmod{N}.$$
- Using *theta–functions* one can, for example, find an explicit formula for the number of ways of expressing n as a sum of k squares.
- *The Congruent Number Problem* asks which rational numbers can be the area of a right triangle with rational sides. This can be studied with doubly periodic complex functions.

6. **Riemann surfaces**.

7. **Modular forms**: At the heart of the proof of Fermat's Last Theorem after 350 years.

3. Justification of the term "restricted structure"

First consider a situation in the real numbers. Let $I \subset \mathbb{R}$ be a nonempty connected open set, i.e., an open interval.

Then each of the following containments is proper:
$$\mathcal{C}^0(I) \supset \mathcal{D}^1(I)$$
(there exist everywhere-continuous, nowhere-differentiable functions),
$$\mathcal{D}^1(I) \supset \mathcal{C}^1(I)$$
(the example f given earlier here, after the uniqueness theorem, is an example),
$$\mathcal{C}^1(I) \supset \mathcal{D}^2(I)$$
(the integral up to x of any example from the containment $\mathcal{C}^0(I) \supset \mathcal{D}^1(I)$ works),
$$\mathcal{D}^2(I) \supset \mathcal{C}^2(I)$$
(the integral up to x of any example from the containment $\mathcal{D}^1(I) \supset \mathcal{C}^1(I)$ works), and onward, down to
$$\mathcal{C}^n(I) \supset \mathcal{C}^\infty(I)$$
and
$$\mathcal{C}^\infty(I) \supset \mathcal{C}^\omega(I)$$
(the function $f(x) = e^{-1/x^2}$ works, extended to $f(0) = 0$). [The "exponential growth dominates polynomial growth" handout is relevant here.]

The situation is messy: there are fewer objects at each new level of structure.

Now consider the analogous situation in the complex numbers. Let $\Omega \subset \mathbb{C}$ be a nonempty connected open set, i.e., a region. Then it turns out that

$$\mathcal{D}^1(\Omega) = \mathcal{C}^\omega(\Omega).$$

All the proper containments collapse. One derivative, not even known to be continuous, implies infinitely many derivatives and power series representation!

4. The basic techniques

1. **The complex number system** $\mathbb{C}$. Geometrically and topologically this has the structure of a plane; algebraically this is a complete, algebraically closed field. These varous descriptions of $\mathbb{C}$ interact in rich ways.

2. **Path integration**. Suppose

$$f : \Omega \longrightarrow \mathbb{C}$$

is a complex–differentiable function on a region, and

$$\gamma : I \longrightarrow \Omega$$

is a *simple closed curve*, i.e., a path that never crosses itself until it ends where it starts, and suppose that γ doesn't go around any holes in Ω. Then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta \qquad \text{for all } z \text{ inside } \gamma.$$

This is *Cauchy's integral representation of* f.

It is exciting for two reasons: First, it shows that that the values of f *on* the curve γ determine the values of f *everywhere inside* γ. Second, it suggests the possibility of differentiating under the integeral sign to get

$$f'(z) = \frac{1}{2\pi i}\frac{d}{dz}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

3. **Power series.** These are easy to manipulate termwise like polynomials. For example, infinite long division (or an infinite system of equations) gives

$$\begin{aligned}\frac{X}{e^X - 1} &= \frac{X}{X + X^2/2! + X^3/3! + \cdots} \\ &= 1 - \frac{1}{2}X + \frac{1}{6}\frac{X^2}{2!} - \frac{1}{30}\frac{X^4}{4!} + \frac{1}{42}\frac{X^6}{6!} \\ &\qquad - \frac{1}{30}\frac{X^8}{8!} + \frac{5}{66}\frac{X^{10}}{10!} - \frac{691}{2730}\frac{X^{12}}{12!} + \cdots\end{aligned}$$

The coefficients here are the *Bernoulli numbers.* An essentially identical way to compute them is by writing

$$\begin{aligned}X &= (e^X - 1)\sum_{n=0}^{\infty}\frac{B_n}{n!}X^n \\ &= \left(\frac{1}{1!}X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots\right)\left(\frac{B_0}{0!} + \frac{B_1}{1!}X + \frac{B_2}{2!}X^2 + \cdots\right),\end{aligned}$$

so that matching coefficients gives

$$1 = B_0,$$
$$0 = B_1 + B_0/2! \quad \text{(solve for } B_1 \text{ since we know } B_0\text{)},$$
$$0 = B_2/2! + B_1/2! + B_0/3! \quad \text{(solve for } B_2 \text{ since we know } B_1 \text{ and } B_0\text{)},$$

and so on.

5. Further Techniques

4. **Beginning real analysis.** Compactness and uniformity underlie some of the required manipulations with integrals and infinite sums. If you are already familiar with these ideas, it is satisfying to see them applied; if they are new to you, seeing them introduced in a specific context can suggest their general importance.

5. **Beginning modern algebra.** Matrix groups, group actions, projective coordinates, and algebraic curves arise naturally in the theory.

6. **Beginning topology.** Results such as the Jordan curve theorem, the winding number, homotopy, homology fit into the correct formulations of the results on path integration. Covering maps figure in Riemann surface theory and the proof of the Picard Theorem.