MATHEMATICS 311: COMPLEX ANALYSIS — ASSIGNMENT 5

Reading: Marsden, sections 3.1, 3.2.

Problems: 1. The Bernoulli numbers B_n are defined to be the coefficients of the power series expansion

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

(a) Show that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$. (Hint: Expand the double power series $z = (e^z - 1) \sum (B_n/n!) z^n$.)

(b) Show that in general these numbers satisfy the recurrence formula

$$\binom{n}{0} B_0 + \binom{n}{1} B_1 + \binom{n}{2} B_2 + \cdots + \binom{n}{n-1} B_{n-1} = 0, \quad (n > 1),$$

so they are all rational numbers. Note that this formula can be written symbolically $(B+1)^n = B^n$, for expanding the left–hand side by the binomial formula then replacing B^k by B_k yields the recurrence. (Hint: Rearrange the double sum in (a).)

(c) Show that

$$\frac{z}{e^z - 1} + \frac{z}{2} = \frac{z}{2} \frac{e^z + 1}{e^z - 1},$$

note that this is an even function and conclude that $B_n = 0$ whenever n is odd and greater than 1.

(d) Show by using the preceding formula that

$$z \cot z = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}.$$

(e) Show by using the preceding formula and the formula $\tan z = \cot z - 2 \cot 2z$ that

$$\tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n} - 1) B_{2n}}{(2n)!} z^{2n-1}.$$

The notation for the Bernoulli numbers differs somewhat, so be careful when comparing formulas from various sources. The Bernoulli numbers vary rather mysteriously and erratically; for your information,

$$B_4 = -1/30, \quad B_6 = 1/42, \quad B_8 = -1/30, \quad B_{10} = 5/66,$$
$$B_{12} = -691/2730, \quad B_{14} = 7/6, \quad B_{20} = -174611/330,$$
$$B_{30} = 8615841276005/14322.$$

2. Derive the power series expansions

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad \log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

and find the radius of convergence of each.

3. Let $f(z)$ be the analytic function in the disk $D = \{z : |z| < 1\}$ for which $f'(z) = 1/(1+z^2)$ and $f(0) = 0$. From the power series expansion

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

deduce the power series expansion

$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}$$

and indicate why your deduction is justified. What is the radius of convergence of this series? Show that

$$\frac{d}{dz}(f(\tan z)) = 1 \text{ and } f(\tan z) = z$$

in a disk about the origin, so that $f(z)$ may be viewed as the inverse tangent function there.