MATHEMATICS 311: COMPLEX ANALYSIS — ASSIGNMENT 1

Reading: Marsden, sections 1.1, 1.2, 1.4.

Problems:

1. Show that $\mathbb{C}$, the field of complex numbers, is isomorphic to the set

$$S = \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix} : x, y \in \mathbb{R} \right\}$$

with the usual matrix addition and multiplication. (An isomorphism is an operation-preserving bijection.)

2. (a) Find the two values of $\sqrt{-9i}$; (b) Find the four values of $\sqrt[4]{-16}$. Express your answers (though not your intermediate work) in Cartesian form.

3. A valid position of minute and hour hands on a clock is called *exchangeable* if swapping the hands yields another valid clock position. Find all exchangeable clock positions. (The idea is to use complex numbers, of course.)

4. Show that if a, b are complex numbers with $b \neq 0$ then $a + \overline{a} + b\overline{z} + \overline{b}z = 0$ is the equation of a line in the plane, and any line in the plane is described by such an equation. (The equation of a line is $A + Bx + Cy = 0$ where not both B and C are zero.)

5. Show that all roots of the polynomial $z^8 + 2iz^5 + 4$ lie outside the unit circle. (The idea is to use the Triangle Inequality.)

6. By evaluating the product $(2+i)(3+i)$ and then viewing the result geometrically, show that

$$\pi/4 = \arctan 1/2 + \arctan 1/3.$$

7. Any three complex numbers can be taken as the vertices of triangle, so one can refer to such a triangle as the triangle (z_1, z_2, z_3).

(a) Show that the triangles (z_1, z_2, z_3) and (w_1, w_2, w_3) are similar with z_j corresponding to w_j for $j = 1, 2, 3$ if and only if

$$\det \begin{bmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = 0.$$

(Hint: both conditions hold if and only if some affine map $z \mapsto az + b$ takes $z_j \mapsto w_j$ for $j = 1, 2, 3$.)

(b) Use part (a) to show that the triangle (z_1, z_2, z_3) is equilateral if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$