HEISENBERG UNCERTAINTY

Consider a wavefunction

$$f : \mathbb{R} \longrightarrow \mathbb{C}.$$

Assume that f is well-behaved, so for example f might be a Schwartz function. For future convenience, introduce the notation $\langle\ ,\ \rangle$ for the inner product, i.e.,

$$\langle f, g\rangle \stackrel{\text{def}}{=} \int f(x)\overline{g(x)}\,dx \qquad \text{for suitable functions } f \text{ and } g$$

(all integrals here are taken over $\mathbb{R}$), and similarly for the L^2-norm,

$$\|f\|_2 \stackrel{\text{def}}{=} \sqrt{\langle f, f\rangle} = \sqrt{\int |f(x)|^2\,dx} \qquad \text{for suitable functions } f.$$

In particular, the L^2-norm of f is 1,

$$\|f\|_2 = 1.$$

Consider the operators

$$A = i\frac{d}{dx}\,, \qquad B = x \text{ (multiplication by } x).$$

Their Lie bracket is

$$(AB - BA)f = i\frac{d}{dx}(xf) - ix\frac{d}{dx}f = if + ixf' - ixf' = if.$$

Or, more concisely, as a relation of operators,

$$-i[A, B] = \mathrm{id}.$$

And in fact, all that we need is the weaker relation that A and B are a *complementary pair* of operators,

$$-i[A, B] \geq \mathrm{id}, \quad \text{meaning that } -i\langle [A, B]f, f\rangle \geq \langle f, f\rangle \text{ for all suitable } f.$$

Also, the operators A and B are self-adjoint because for suitable functions f and g,

$$\langle Af, g\rangle = \int if'\,\overline{g} = i\int f'\,\overline{g} = -i\int f\,\overline{g}' = \int f\,\overline{ig'} = \langle f, Ag\rangle,$$

and

$$\langle Bf, g\rangle = \int xf\overline{g} = \int f\overline{xg} = \langle f, Bg\rangle.$$

Now compute with the wavefunction as follows:

$$1 = \langle f, f\rangle \leq -i\langle (AB - BA)f, f\rangle = -i\left(\langle Bf, Af\rangle - \langle Af, Bf\rangle\right) = -2\,\mathrm{Im}\left(\langle Af, Bf\rangle\right).$$

So in fact we have, using the Cauchy–Schwarz inequality for the second step,

$$1 \leq 2\,|\langle Af, Bf\rangle| \leq 2\,\|Af\|_2 \cdot \|Bf\|_2 = 2\,\|f'(x)\|_2 \cdot \|xf(x)\|_2.$$

Recall that the Fourier transform is an isometry that converts differentiation to multiplication by the variable,

$$\|f'(x)\|_2 = \|\widehat{f'}(\xi)\|_2 = 2\pi\,\|\xi\widehat{f}(\xi)\|_2.$$

Consequently

$$1 \leq 4\pi \, \|xf(x)\|_2 \cdot \|\xi \widehat{f}(\xi)\|_2.$$

Thus the product of the spreads of f and $\widehat{f}$ is bounded from below. Since f is a wavefunction, its Fourier transform $\widehat{f}$ is momentum. This is the Heisenberg principle, normalized to the case of functions whose A-means and B-means are zero. The general case follows from a change of variables.