GAMMA FUNCTION SYMMETRY AND DUPLICATION

In the open right complex half plane, the *gamma function* is

$$\Gamma(s) = \int_{t=0}^{\infty} t^s e^{-t}\, \frac{\mathrm{d}t}{t}, \quad \mathrm{Re}(s) > 0.$$

Two basic properties of gamma are

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.
- $\Gamma(s+1) = s\Gamma(s)$, so that $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$.

The volume of the n-dimensional unit ball is $\pi^{n/2}/(n/2)!$ for $n = 1, 2, 3, \ldots$, where naturally $(n/2)!$ is understood to mean $\Gamma(n/2+1)$.

Various methods extend the gamma function meromorphically to the full complex plane. One approach is to note that the left side of the equality

$$\Gamma(s+1) = s\Gamma(s)$$

is defined on the larger half plane $\mathrm{Re}(s) > -1$, defining the right side on the larger half plane as well; now the left side is defined on $\mathrm{Re}(s) > -2$, and so on.

A second approach is to note that the integral $\int_{t=0}^{\infty} t^s e^{-t}\, \mathrm{d}t/t$ converges robustly for all complex s at its upper endpoint and is fragile only at its lower endpoint, requiring $\mathrm{Re}(s) > 0$ there. Thus, for $\mathrm{Re}(s) > 0$ we break the integral into two pieces and then pass the exponential power series through the first one,

$$\begin{aligned}
\Gamma(s) &= \int_{t=0}^{1} t^s e^{-t}\, \frac{\mathrm{d}t}{t} + \int_{t=1}^{\infty} t^s e^{-t}\, \frac{\mathrm{d}t}{t} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{t=0}^{1} t^{s+n}\, \frac{\mathrm{d}t}{t} + \int_{t=1}^{\infty} t^s e^{-t}\, \frac{\mathrm{d}t}{t} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(s+n)} + \int_{t=1}^{\infty} t^s e^{-t}\, \frac{\mathrm{d}t}{t}.
\end{aligned}$$

The last expression just computed extends meromorphically to $\mathbb{C}$, with a simple pole at each nonpositive integer $-n$, where the residue is $(-1)^n/n!$.

A third approach is suggested by the second one, as follows. Because $\Gamma(s)$ has a simple pole at each nonpositive integer as just described, $\Gamma(s)\Gamma(1-s)$ has a simple pole at every integer. Further the residue of $\Gamma(s)\Gamma(1-s)$ at any nonpositive integer $-n$ is $(-1)^n$ because $\Gamma(n+1) = n!$. And because $\Gamma(s)\Gamma(1-s)$ is symmetric about the vertical line $\mathrm{Re}(s) = 1/2$, similarly its residue at any positive integer n is also $(-1)^n$. Beyond this, we have

$$\Gamma(s)\Gamma(1-s) = \frac{\Gamma(s+1)}{s}(1-s-1)\Gamma(1-s-1) = -\Gamma(s+1)\Gamma(1-(s+1)),$$

so that $\Gamma(s)\Gamma(1-s)$ has skew-period 1. All these properties of $\Gamma(s)\Gamma(1-s)$ are shared by the function $\pi/\sin \pi s$, and so we wonder how closely the two are related.

In fact they are equal. It suffices to show that

$$\boxed{\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < \mathrm{Re}(s) < 1.}$$

And then this identity can be used to extend $\Gamma(s)$ meromorphically to $\mathbb{C}$ without reference to the arguments given above. With these ideas in mind, this writeup establishes the boxed identity.

The *Haar measure* of the multiplicative group of positive real numbers $(\mathbb{R}^\times_{>0}, \cdot)$ is

$$\mathrm{d}\mu(t) = \frac{\mathrm{d}t}{t}.$$

Compatibly with the familiar rules $\mathrm{d}(t+c) = \mathrm{d}t$ and $\mathrm{d}(at) = a\,\mathrm{d}t$ for the usual measure $\mathrm{d}t$ of the additive group $(\mathbb{R}, +)$, we have

$$\mathrm{d}\mu(ct) = \frac{\mathrm{d}(ct)}{ct} = \frac{\mathrm{d}t}{t} = \mathrm{d}\mu(t).$$

and

$$\mathrm{d}\mu(t^a) = \frac{\mathrm{d}(t^a)}{t^a} = a\frac{\mathrm{d}t}{t} = a\,\mathrm{d}\mu(t).$$

The integral $\int_{t=1}^{\infty} t^s\,\mathrm{d}\mu(t)$ converges for $\mathrm{Re}(s) < 0$, and so, because $\mathrm{d}\mu(t^{-1}) = -\mathrm{d}\mu(t)$, the integral $\int_{t=0}^{1} t^s\,\mathrm{d}\mu(t)$ converges for $\mathrm{Re}(s) > 0$.

The definition of the gamma function as an integral is really

$$\Gamma(s) = \int_{\mathbb{R}^\times_{>0}} t^s e^{-t}\,\mathrm{d}\mu(t), \quad \mathrm{Re}(s) > 0.$$

In the usual notation for the gamma integral as in integral from 0 to ∞, it should be understood that the lower limit of integration 0 is just as improper as the upper limit ∞. Despite the lower limit of integration being improper, the integral converges for $\mathrm{Re}(s) > 0$, as just explained. Also, the gamma integral converges at its improper upper limit of integration because the exponential decay of e^{-t} dominates the polynomial growth of t^s.

Now we establish the desired identity,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < \mathrm{Re}(s) < 1.$$

To do so, it suffices to consider only *real* s between 0 and 1. For such s, the definition of gamma gives

$$\Gamma(s)\Gamma(1-s) = \iint_{\mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{>0}} w^s x^{1-s} e^{-w-x}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(w).$$

Replace x by wx and recall that $\mathrm{d}\mu(wx) = \mathrm{d}\mu(x)$,

$$\Gamma(s)\Gamma(1-s) = \iint_{\mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{>0}} w x^{1-s} e^{-w(x+1)}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(w).$$

Exchange the order of integration and change to ordinary measure,

$$\Gamma(s)\Gamma(1-s) = \int_{x=0}^{\infty} x^{-s} \int_{w=0}^{\infty} e^{-(x+1)w}\,\mathrm{d}w\,\mathrm{d}x.$$

The inner integral is $1/(x+1)$, leaving

$$\Gamma(s)\Gamma(1-s) = \int_{x=0}^{\infty} \frac{x^{-s}\,\mathrm{d}x}{x+1}.$$

And we have evaluated this last integral by contour integration,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < s < 1.$$

As above, the result extends by uniqueness to all complex s such that $0 < \mathrm{Re}(s) < 1$, and then it extends Γ to all of $\mathbb{C}$.

The relation $\Gamma(s)\Gamma(1-s) = \int_{x=0}^{\infty} x^{-s}/(x+1)\,\mathrm{d}x$ for $0 < s < 1$ is a special case of the more general relation $\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b)$ for $a, b > 0$, where B is the beta integral $B(a,b) = \int_{x=0}^{1} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$. The more general relation gives a proof of the Legendre duplication formula for the gamma function. We explain these matters next.

The beta function is

$$B(a,b) = \int_{x=0}^{1} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x, \quad a > 0,\ b > 0.$$

Compute, with $x = \frac{1+y}{2}$ at the second step to follow and then with $x = y^2$ so that $\mathrm{d}x = 2y\,\mathrm{d}y$ and thus $\mathrm{d}y = \frac{1}{2x^{1/2}}\,\mathrm{d}x = \frac{1}{2}x^{1/2-1}\,\mathrm{d}x$ at the fifth step, that for $b > 0$,

$$\begin{aligned}
B(b,b) &= \int_{x=0}^{1} \big(x(1-x)\big)^{b-1}\,\mathrm{d}x \\
&= \frac{1}{2}\int_{y=-1}^{1} \left(\frac{1+y}{2}\cdot\frac{1-y}{2}\right)^{b-1} dy \\
&= 2^{1-2b}\int_{y=-1}^{1} (1-y^2)^{b-1}\,dy \\
&= 2^{2-2b}\int_{y=0}^{1} (1-y^2)^{b-1}\,dy \\
&= 2^{1-2b}\int_{x=0}^{1} x^{1/2-1}(1-x)^{b-1}\,dx \\
&= 2^{1-2b}B(\tfrac{1}{2},b).
\end{aligned}$$

Repeating,

$$B(b,b) = 2^{1-2b}B(\tfrac{1}{2},b), \quad b > 0. \tag{1}$$

Also, we will show below that

$$\Gamma(a)\Gamma(b) = \Gamma(a+b)B(a,b), \quad a > 0,\ b > 0. \tag{2}$$

It follows that for all $s > 0$,

$$\begin{aligned}
\Gamma(\tfrac{s}{2})^2 &= \Gamma(s)B(\tfrac{s}{2},\tfrac{s}{2}) && \text{by (2) with } a = b = \tfrac{s}{2} \\
&= \Gamma(s)2^{1-s}B(\tfrac{1}{2},\tfrac{s}{2}) && \text{by (1) with } b = \tfrac{s}{2} \\
&= \Gamma(s)2^{1-s}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{s}{2})}{\Gamma(\frac{s+1}{2})} && \text{by (2) with } a = \tfrac{1}{2},\ b = \tfrac{s}{2}.
\end{aligned}$$

Because $\Gamma(\frac{1}{2}) = \pi^{1/2}$, this gives Legendre's formula $\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = 2^{1-s}\pi^{1/2}\Gamma(s)$ for $s > 0$. And because $\Gamma(s)/\big(\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})\big)$ is entire, this relation extends meromorphically to the full complex plane,

$$\boxed{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\pi^{1/2}\Gamma(s), \quad s \in \mathbb{C}.}$$

To complete the argument, we establish (2). Compute for $a, b > 0$, using Fubini's theorem and the Haar measure property $\mathrm{d}(cz)/(cz) = \mathrm{d}z/z$ freely, that

$$\begin{aligned}
\Gamma(a)\Gamma(b) &= \int_{t>0} e^{-t} t^a \,\frac{\mathrm{d}t}{t} \int_{u>0} e^{-u} u^b \,\frac{\mathrm{d}u}{u} \\
&= \int_{t>0} \int_{u>0} e^{-t-u} t^a u^b \,\frac{\mathrm{d}u}{u}\,\frac{\mathrm{d}t}{t} \\
&= \int_{t>0} \int_{u>0} e^{-t-tu} t^a (tu)^b \,\frac{\mathrm{d}u}{u}\,\frac{\mathrm{d}t}{t} \\
&= \int_{u>0} \int_{t>0} e^{-(1+u)t} t^{a+b} \,\frac{\mathrm{d}t}{t}\, u^b \,\frac{\mathrm{d}u}{u} \\
&= \int_{u>0} \int_{t>0} e^{-t} \left(\frac{t}{1+u}\right)^{a+b} \frac{\mathrm{d}t}{t}\, u^b \,\frac{\mathrm{d}u}{u} \\
&= \int_{t>0} e^{-t} t^{a+b} \,\frac{\mathrm{d}t}{t} \int_{u>0} \left(\frac{1}{1+u}\right)^{a+b} u^b \,\frac{\mathrm{d}u}{u} \\
&= \Gamma(a+b) \int_{u>0} \left(\frac{1}{1+u}\right)^{a+1} \left(\frac{u}{1+u}\right)^{b-1} \mathrm{d}u.
\end{aligned}$$

Let $x = 1/(1+u)$, so that x goes from 1 to 0 and $\mathrm{d}u = \mathrm{d}(1/x - 1) = -\mathrm{d}x/x^2$, to get the desired result,

$$\Gamma(a)\Gamma(b) = \Gamma(a+b) \int_{x=0}^{1} x^{a-1}(1-x)^{b-1}\,\mathrm{d}x = \Gamma(a+b)B(a,b).$$

As an end note, we observe that the methods here again establish the symmetry formula for the gamma function. Specifically, for $0 < s < 1$, the long computation just shown also gives, with $a = s$ and $b = 1 - s$, now denoting the variable of integration x rather than u,

$$\Gamma(s)\Gamma(1-s) = \Gamma(1) \int_{x>0} \left(\frac{1}{1+x}\right)^{s+1} \left(\frac{x}{1+x}\right)^{-s} \mathrm{d}x = \int_{x>0} \frac{x^{-s}}{1+x}\,\mathrm{d}x.$$

We have evaluated this last integral by contour integration and then noted that the resulting identity extends to all s,

$$\boxed{\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}\,, \quad s \in \mathbb{C}.}$$