SYMMETRY OF THE GAMMA FUNCTION

In the open right complex half plane, the *gamma function* is

$$\Gamma(s) = \int_{t=0}^{\infty} t^s e^{-t} \, \frac{\mathrm{d}t}{t}, \quad \mathrm{Re}(s) > 0.$$

Two basic properties of gamma are

- $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$.
- $\Gamma(s+1) = s\Gamma(s)$, so that $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \cdots$.

The volume of the n-dimensional unit ball is $\pi^{n/2}/(n/2)!$ for $n = 1, 2, 3, \cdots$, where naturally $(n/2)!$ is understood to mean $\Gamma(n/2+1)$.

Various methods extend the gamma function meromorphically to the full complex plane. One approach is to note that the left side of the equality

$$\Gamma(s+1) = s\Gamma(s)$$

is defined on the larger half plane $\mathrm{Re}(s) > -1$, defining the right side on the larger half plane as well; now the left side is defined on $\mathrm{Re}(s) > -2$, and so on.

A second approach is to note that the integral $\int_{t=0}^{\infty} t^s e^{-t} \, \mathrm{d}t/t$ converges robustly for all complex s at its upper endpoint and is fragile only at its lower endpoint, requiring $\mathrm{Re}(s) > 0$ there. Thus, for $\mathrm{Re}(s) > 0$ we break the integral into two pieces and then pass the exponential power series through the first one,

$$\begin{aligned}
\Gamma(s) &= \int_{t=0}^{1} t^s e^{-t} \, \frac{\mathrm{d}t}{t} + \int_{t=1}^{\infty} t^s e^{-t} \, \frac{\mathrm{d}t}{t} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{t=0}^{1} t^{s+n} \, \frac{\mathrm{d}t}{t} + \int_{t=1}^{\infty} t^s e^{-t} \, \frac{\mathrm{d}t}{t} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(s+n)} + \int_{t=1}^{\infty} t^s e^{-t} \, \frac{\mathrm{d}t}{t}.
\end{aligned}$$

The last expression just computed extends meromorphically to $\mathbb{C}$, with a simple pole at each nonpositive integer $-n$, where the residue is $(-1)^n/n!$.

A third approach is suggested by the second one, as follows. Because $\Gamma(s)$ has a simple pole at each nonpositive integer as just described, $\Gamma(s)\Gamma(1-s)$ has a simple pole at every integer. Further the residue of $\Gamma(s)\Gamma(1-s)$ at any nonpositive integer $-n$ is $(-1)^n$ because $\Gamma(n+1) = n!$. And because $\Gamma(s)\Gamma(1-s)$ is symmetric about the vertical line $\mathrm{Re}(s) = 1/2$, similarly its residue at any positive integer n is also $(-1)^n$. All these properties of $\Gamma(s)\Gamma(1-s)$ are shared by the function $\pi/\sin \pi s$, and so we wonder how the two are related.

In fact they are equal. It suffices to show that

$$\boxed{\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < \mathrm{Re}(s) < 1.}$$

And then this identity can be used to extend $\Gamma(s)$ meromorphically to $\mathbb{C}$ without reference to the arguments given above. With these ideas in mind, this writeup establishes the boxed identity.

The *Haar measure* of the multiplicative group of positive real numbers $(\mathbb{R}^\times_{>0}, \cdot)$ is

$$\mathrm{d}\mu(t) = \frac{\mathrm{d}t}{t}\,.$$

Compatibly with the familiar rules $\mathrm{d}(t+c) = \mathrm{d}t$ and $\mathrm{d}(at) = a\,\mathrm{d}t$ for the usual measure $\mathrm{d}t$ of the additive group $(\mathbb{R}, +)$, we have

$$\mathrm{d}\mu(ct) = \frac{\mathrm{d}(ct)}{ct} = \frac{\mathrm{d}t}{t} = \mathrm{d}\mu(t).$$

and

$$\mathrm{d}\mu(t^a) = \frac{\mathrm{d}(t^a)}{t^a} = a\frac{\mathrm{d}t}{t} = a\,\mathrm{d}\mu(t).$$

The integral $\int_{t=1}^\infty t^s\,\mathrm{d}\mu(t)$ converges for $\mathrm{Re}(s) < 0$, and so, because $\mathrm{d}\mu(t^{-1}) = -\mathrm{d}\mu(t)$, the integral $\int_{t=0}^1 t^s\,\mathrm{d}\mu(t)$ converges for $\mathrm{Re}(s) > 0$.

The definition of the gamma function as an integral is really

$$\Gamma(s) = \int_{\mathbb{R}^\times_{>0}} t^s e^{-t}\,\mathrm{d}\mu(t), \quad \mathrm{Re}(s) > 0.$$

In the usual notation for the gamma integral as in integral from 0 to ∞, it should be understood that the lower limit of integration 0 is just as improper as the upper limit ∞. Despite the lower limit of integration being improper, the integral converges for $\mathrm{Re}(s) > 0$, as just explained. Also, the gamma integral converges at its improper upper limit of integration because the exponential decay of e^{-t} dominates the polynomial growth of t^s.

Now we establish the desired identity,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < \mathrm{Re}(s) < 1.$$

To do so, it suffices to consider only *real* s between 0 and 1. For such s, the definition of gamma gives

$$\Gamma(s)\Gamma(1-s) = \iint_{\mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{>0}} w^s x^{1-s} e^{-w-x}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(w).$$

Replace x by wx and recall that $\mathrm{d}\mu(wx) = \mathrm{d}\mu(x)$,

$$\Gamma(s)\Gamma(1-s) = \iint_{\mathbb{R}^\times_{>0} \times \mathbb{R}^\times_{>0}} w x^{1-s} e^{-w(1+x)}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(w).$$

Exchange the order of integration and change to ordinary measure,

$$\Gamma(s)\Gamma(1-s) = \int_{x=0}^\infty x^{-s} \int_{w=0}^\infty e^{-(1+x)w}\,\mathrm{d}w\,\mathrm{d}x.$$

The inner integral is $1/(1+x)$, leaving

$$\Gamma(s)\Gamma(1-s) = \int_{x=0}^\infty \frac{x^{-s}\,\mathrm{d}x}{1+x}\,.$$

And we have evaluated this last integral by contour integration,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \quad 0 < s < 1.$$

As above, the result extends by uniqueness to all complex s such that $0 < \mathrm{Re}(s) < 1$, and then it extends Γ to all of $\mathbb{C}$.