DIRICHLET'S PROBLEM ON THE DISK

Let D denote the unit disk, and let $\overline{D}$ denote its closure. Dirichlet's Problem is: *Given a bounded, piecewise continuous function on the boundary of the disk,*

$$u : \partial D \longrightarrow \mathbb{R},$$

is there a harmonic function on the disk,

$$u : D \longrightarrow \mathbb{R},$$

that extends to match the original u on the boundary of the disk, the extension being continuous other than the original discontinuities?

This problem is readily solved with the techniques of complex analysis. On a simply connected domain, a harmonic function u and an analytic function f are essentially the same thing via the formula

$$f = u + iv, \quad v \text{ a harmonic conjugate of } u.$$

Combining this with Cauchy's integral formula for analytic functions f gives *Poisson's integral formula* for harmonic functions u, expressing the values of u on a disk entirely in terms of the values of u on the boundary circle. Then the idea is to turn Poisson's formula around and use it to *define* the values of u on the disk in terms of the values on the circle. That is, like many mathematical arguments, we show that a set of conditions is sufficient for some purpose by first assuming whatever we want them to imply, then deriving necessary consequences therefrom, and finally using these consequences to argue that the given conditions are indeed sufficient, as we wanted them to be.

With this outline in mind, start by assuming that we *have* a harmonic function

$$u : \overline{D} \longrightarrow \mathbb{R},$$

even though our eventual goal is to *find* one. More precisely, u is defined on an open superset Ω of $\overline{D}$, u is twice continuously differentiable, and the Laplacian of u is zero. We may take Ω to be simply connected. A harmonic conjugate of u is

$$v : \Omega \longrightarrow \mathbb{R}, \qquad v(z) = \int_0^z (u_x \, \mathrm{d}y - u_y \, \mathrm{d}x),$$

and the corresponding complex-valued function built from u and v is analytic,

$$f : \Omega \longrightarrow \mathbb{C}, \qquad f = u + iv.$$

By Cauchy's integral formula,

$$f(0) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta}.$$

Note that $\frac{\mathrm{d}\zeta}{i\zeta} = \mathrm{d}(\arg \zeta)$ as ζ traverses the circle. Thus, taking the real part in the previous display gives the *mean value property of harmonic functions*,

$$u(0) = \frac{1}{2\pi} \int_{|\zeta|=1} u(\zeta)\,\mathrm{d}(\arg \zeta) = \frac{1}{2\pi} \int_{|\zeta|=1} u(\zeta)\,\frac{\mathrm{d}\zeta}{i\zeta}\,.$$

This formula is not particular to the unit disk: the value of a harmonic function at any point is the average of its values over any circle about the point, so long as the circle and its interior lie in the domain of the function.

(Also, the mean value property of harmonic functions does not depend on complex analysis despite our use of complex analysis to derive it. Instead one can define the symmetrization of u with respect to angle,

$$w : \overline{D} \longrightarrow \mathbb{R}, \quad w(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} u(ze^{i\theta})\, \mathrm{d}\theta.$$

Since $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$, it follows that $r^2 w_{rr} + r w_r = 0$. Standard differential equation techniques show that $w(r) = a + b \log r$. But w is bounded near 0, so w is constant, and the constant must be $u(0)$.)

We want a similar integral formula for values $u(z)$ for all $z \in D$, not just $z = 0$. The idea is to move z to 0 and keep track of how this affects the mean value formula that we already have. For any $z \in D$, recall the standard automorphism of D that moves z to 0, the fractional linear transformation

$$T_z = \begin{bmatrix} 1 & -z \\ -\overline{z} & 1 \end{bmatrix} : \zeta \longmapsto \frac{\zeta - z}{1 - \overline{z}\zeta}\,.$$

The composition $u \circ T_z^{-1}$ is again harmonic, and

$$u(z) = (u \circ T_z^{-1})(0).$$

Thus, by the mean value property of harmonic functions,

$$u(z) = \frac{1}{2\pi} \int_{|\zeta|=1} u(T_z^{-1}\zeta)\, \frac{\mathrm{d}\zeta}{i\zeta} = \frac{1}{2\pi} \int_{|\xi|=1} u(\xi)\, \frac{\mathrm{d}T_z\xi}{i\,T_z\xi}.$$

In general, the derivative of a fractional linear transformation is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}'(\xi) = \frac{ad - bc}{(c\xi + d)^2}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{C}).$$

It follows by a short calculation that

$$\frac{\mathrm{d}T_z\xi}{i\,T_z\xi} = \frac{1 - |z|^2}{|\xi - z|^2}\, \frac{\mathrm{d}\xi}{i\xi}.$$

That is, letting $\xi = e^{i\theta}$, so that $\frac{\mathrm{d}\xi}{i\xi} = d\theta$,

$$u(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} u(e^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2}\, \mathrm{d}\theta.$$

This is Poisson's formula. Also it can be written

$$u(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} u(e^{i\theta})\, \mathrm{Re}\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) \mathrm{d}\theta\,,$$

and by the Law of Cosines it is further (with $z = r_z e^{i\theta_z}$)

$$u(r_z e^{i\theta_z}) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} u(e^{i\theta}) \frac{1 - r_z^2}{1 - 2r_z \cos(\theta - \theta_z) + r_z^2}\, \mathrm{d}\theta.$$

The quantity against which $u(e^{i\theta})$ is being integrated, which has the three expressions shown here, is the *Poisson kernel*. That is,

$$u(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} u(e^{i\theta}) K_z(\theta)\, \mathrm{d}\theta\,,$$

where

$$K_z(\theta) = \frac{1-|z|^2}{|e^{i\theta}-z|^2} = \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) = \frac{1-r_z^2}{1-2r_z\cos(\theta-\theta_z)+r_z^2}.$$

Poisson's formula has the pleasing physical interpretation that the value $u(z)$ for z in the disk is determined by the values of $u(e^{i\theta})$ for $e^{i\theta}$ on the circle according to an inverse square law weighting. Note that in particular, letting u be identically 1 gives

$$\frac{1}{2\pi}\int_{\theta=0}^{2\pi} K_z(\theta)\,\mathrm{d}\theta = 1, \quad z \in D. \tag{1}$$

We will refer back to this formula soon.

Poisson's formula holds under the weaker hypotheses that u is harmonic on the open disk D and continuous on the closed disk $\overline{D}$. To see this, consider any r such that $0 < r < 1$, and let $D_{1/r}$ be the open disk about 0 of radius $1/r$, an open superset of D. Scaling $D_{1/r}$ by r gives D, where u is harmonic, so define

$$u_r : D_{1/r} \longrightarrow \mathbb{R}, \quad u_r(z) = u(rz).$$

Then u_r is defined and harmonic on $D_{1/r}$, and hence it meets the conditions for Poisson's formula. That is,

$$u_r(z) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} u_r(e^{i\theta})K_z(\theta)\,\mathrm{d}\theta, \quad z \in D,$$

or

$$u(rz) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} u(re^{i\theta})\frac{1-|z|^2}{|e^{i\theta}-z|^2}\,\mathrm{d}\theta, \quad z \in D.$$

Now let $r \to 1$ to obtain Poisson's formula as before. Passing the limit through the integral is justified as usual by continuity, compactness, and uniformity.

Next, the plan is to turn all of this around. Given a bounded, piecewise continuous function

$$u : \partial D \longrightarrow \mathbb{R},$$

use Poisson's formula to define a function

$$P_u : D \longrightarrow \mathbb{R}$$

where

$$P_u(z) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} u(e^{i\theta})K_z(\theta)\,\mathrm{d}\theta.$$

Because $K_z(\theta) = \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)$, P_u is the real part of the analytic function

$$f(z) = \frac{1}{2\pi i}\int_{|\xi|=1} u(\xi)\,\frac{\xi+z}{\xi-z}\,\frac{\mathrm{d}\xi}{\xi}\,,$$

and so P_u is harmonic.

The question is whether P_u extends continuously to the original function on the boundary of the disk, i.e., whether it is true that

$$\lim_{z\to e^{i\theta_0}} P_u(z) = u(e^{i\theta_0}).$$

The answer, due to Schwarz, is that *at points $e^{i\theta_0}$ where u is continuous, it does.*

To see this, first observe some general properties of the functional

$$u \longmapsto P_u.$$

These are

- (Linearity) $P_{u_1+u_2} = P_{u_1} + P_{u_2}$ and $P_{cu} = cP_u$.
- (Positivity) If $u \geq 0$ on ∂D then $P_u \geq 0$ on D.
- (Preservation of identity) $P_1 = 1$.
- (Preservation of bounds) If $m \leq u \leq M$ on ∂D then $m \leq P_u \leq M$ on D.

The first two are straightforward to show, because integration is linear and the Poisson kernel is positive. The preservation of identity is already in place as the *Note that in particular* formula (1) above. And the preservation of bounds is a consequence of the previous properties.

With these in hand, it is easy to prove Schwarz's Theorem. We may assume that $\theta_0 = 0$ (i.e., $e^{i\theta_0} = 1$) and that $u(1) = 0$. The idea is to break the integral $P_u(z)$ into pieces, each of which is small for its own reasons as $z \to 1$, so that indeed $\lim_{z\to 1} P_u(z) = 0$.

Let $\varepsilon > 0$ be given. Since u is continuous at 1, there is an arc C_1 of ∂D about 1 on which $|u| < \varepsilon/2$. Let $C_2 = \partial D - C_1$, and note that u decomposes as $u = u_1 + u_2$ where

$$u_1 = \begin{cases} u & \text{on } C_1, \\ 0 & \text{on } C_2, \end{cases} \qquad u_2 = \begin{cases} 0 & \text{on } C_1, \\ u & \text{on } C_2. \end{cases}$$

Then $|P_{u_1}| < \varepsilon/2$ since $|u_1| < \varepsilon/2$. Meanwhile, P_{u_2} is an integral over C_2. But the Poisson kernel grows small uniformly on C_2 as $z \to 1$, so eventually $|P_{u_2}(z)| < \varepsilon/2$ as well. Since $P_u = P_{u_1} + P_{u_2}$, this completes the proof.