A SERIES REPRESENTATION OF THE COTANGENT

This writeup establishes an equality of meromorphic functions,

$$\begin{aligned}\pi\cot\pi z &= \frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{z+n}\right)\\ &= \frac{1}{z}+2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2}\,.\end{aligned}$$

The function $\pi\cot\pi z$ (for nonintegers $z\in\mathbb{C}$) is analytic and $\mathbb{Z}$-periodic. Near $z=0$ we have

$$\pi\cot\pi z \sim \pi\frac{1}{\pi z}=\frac{1}{z},$$

so that $\pi\cot\pi z$ is also meromorphic at 0, having a simple pole there with residue 1. By $\mathbb{Z}$-periodicity, the same holds at each integer n. Thus, a naïve first attempt to imitate $\pi\cot\pi z$ by a series is

$$\sum_{n\in\mathbb{Z}}\frac{1}{z-n}.$$

However, the nth term of this series is $\mathcal{O}(1/n)$, so that the series is not even summable. One can fix this problem by modifying the terms to obtain the series

$$\frac{1}{z}+\sum_{n\neq 0}\left(\frac{1}{z-n}+\frac{1}{n}\right).$$

Now the nth term is

$$\frac{1}{z-n}+\frac{1}{n}=\frac{z}{(z-n)n}=\mathcal{O}(\frac{1}{n^2}),$$

and so the new series *is* summable. In fact, this calculation shows that the new series is absolutely summable, so that its terms can be rearranged. In particular, pairing the terms for n and $-n$ gives

$$\begin{aligned}\frac{1}{z-n}+\frac{1}{n}+\frac{1}{z+n}-\frac{1}{n} &= \frac{1}{z-n}+\frac{1}{z+n}\\ &= \frac{2z}{z^2-n^2},\end{aligned}$$

and these are the terms of the series that we began with, in both of its forms. So at least that series converges absolutely for any noninteger $z\in\mathbb{C}$.

The convergence is gained at a cost, but the cost is only apparent. With its terms corrected, when our series is evaluated at z and then at $z+1$ its symmetric partial sums are

$$\varphi_n(z)=\frac{1}{z}+\sum_{1\le|j|\le n}\left(\frac{1}{z-j}+\frac{1}{j}\right)=\sum_{j=-n}^{n}\frac{1}{z-j}$$

and

$$\varphi_n(z+1) = \frac{1}{z+1} + \sum_{1\le|j|\le n}\left(\frac{1}{z+1-j} + \frac{1}{j}\right) = \sum_{j=-n}^{n} \frac{1}{z+1-j}\,,$$

and these are not identical termwise. However, their difference nearly cancels,

$$\varphi_n(z+1) - \varphi_n(z) = \frac{1}{z+n+1} - \frac{1}{z-n}\,,$$

and the remaining two terms go to 0 for any z as n grows. Thus $\lim_n \varphi_n(z)$ is indeed $\mathbb{Z}$-periodic. Below we will give a second proof of its periodicity.

All of this said, the series that this writeup begins with (in either of its forms) is not a Laurent series, and so part of the task here is to show that it defines a meromorphic function at all.

To show that the sum is meromorphic, recall a result from a previous writeup: *Let Ω be a region in $\mathbb{C}$. Consider a sequence of differentiable functions on Ω,*

$$\{\varphi_0, \varphi_1, \varphi_2, \dots\} : \Omega \longrightarrow \mathbb{C}.$$

Suppose that the sequence converges on Ω to a limit function

$$\varphi : \Omega \longrightarrow \mathbb{C}$$

and that the convergence is uniform on compact subsets of Ω. Then

(1) *The limit function φ is differentiable.*
(2) *The sequence $\{\varphi_n'\}$ of derivatives converges on Ω to the derivative φ' of the limit function.*
(3) *This convergence is also uniform on compact subsets of Ω.*

To apply the result here, let $\Omega = \mathbb{C} - \mathbb{Z}$, a region in $\mathbb{C}$. Define

$$\varphi_n : \Omega \longrightarrow \mathbb{C}, \qquad \varphi_n(z) = \frac{1}{z} + \sum_{j=1}^{n}\left(\frac{1}{z-j} + \frac{1}{z+j}\right), \quad n = 0, 1, 2, \dots.$$

This is the sequence of partial sums of

$$\varphi : \Omega \longrightarrow \mathbb{C}, \qquad \varphi(z) = \frac{1}{z} + \sum_{j=1}^{\infty}\left(\frac{1}{z-j} + \frac{1}{z+j}\right).$$

Consider any $z \in \Omega$. For all $j > \sqrt{2}|z|$, the reverse triangle inequality gives

$$|z^2 - j^2| \ge j^2 - |z|^2 > j^2 - j^2/2 = j^2/2,$$

and so

$$\left|\frac{1}{z^2 - j^2}\right| < \frac{2}{j^2}.$$

This shows that the partial sums

$$\varphi_n(z) = \frac{1}{z} + 2z\sum_{j=1}^{n} \frac{1}{z^2 - j^2}$$

converge absolutely. Consequently, they converge to the limit function

$$\varphi(z) = \frac{1}{z} + 2z\sum_{j=1}^{\infty} \frac{1}{z^2 - j^2}.$$

We need to show that the convergence is uniform on compact subsets of Ω. Let K be such a subset, and let $\varepsilon > 0$ be given. There is a uniform bound $b > 0$ on the absolute values $|z|$ for all $z \in K$. Also, there a starting index n_0 such that for any $n > n_0$,

$$\sum_{j=n+1}^{\infty} \frac{1}{j^2} < \frac{\varepsilon}{4b}.$$

Consider any n such that $n > n_0$ and also $n > \sqrt{2}b$. For such n and for all $z \in K$,

$$|\varphi(z) - \varphi_n(z)| = \left| 2z \sum_{j=n+1}^{\infty} \frac{1}{z^2 - j^2} \right| \leq 2b \sum_{j=n+1}^{\infty} \left| \frac{1}{z^2 - j^2} \right| \leq 2b \sum_{j=n+1}^{\infty} \frac{2}{j^2} < \varepsilon.$$

This shows that the convergence of $\{\varphi_n\}$ to φ on Ω is uniform on compact subsets.

By the result, the limit function can be differentiated termwise. Now that we no longer need the symbol n to index partial sums, we return to the more natural notation of using it as sum-index,

$$\varphi(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2},$$

and

$$\varphi'(z) = -\sum_{n \in \mathbb{Z}} \frac{1}{(z-n)^2}.$$

The second series for φ shows that it is odd, and the series for φ' shows that it is even. The convergence of φ' is again absolute, and so φ' is $\mathbb{Z}$-periodic by a calculation that rearranges terms,

$$\varphi'(z+m) = -\sum_{n \in \mathbb{Z}} \frac{1}{(z+m-n)^2} = -\sum_{n' \in \mathbb{Z}} \frac{1}{(z-n')^2} \quad \text{where } n' = n - m.$$

It follows that

$$\big(\varphi(z+1) - \varphi(z)\big)' = \varphi'(z+1) - \varphi'(z) = \varphi'(z) - \varphi'(z) = 0,$$

so that

$$\varphi(z+1) - \varphi(z) = c \quad \text{for some constant } c.$$

To show (for a second time) that φ is $\mathbb{Z}$-periodic, we need to show that $c = 0$. But in particular,

$$c = \varphi(1/2) - \varphi(-1/2) = 2\varphi(1/2) \quad \text{since } \varphi \text{ is odd},$$

and so it suffices to show that $\varphi(1/2) = 0$. Inspect it,

$$\varphi(1/2) = 2 + \sum_{n=1}^{\infty} \frac{1}{\frac{1}{4} - n^2} = 2 - \sum_{n=1}^{\infty} \left(\frac{1}{n - \frac{1}{2}} - \frac{1}{n + \frac{1}{2}} \right).$$

The sum telescopes to 2, giving the desired result.

The argument so far shows that the function $\varphi(z) - 1/z$ is also analytic at $z = 0$. Therefore φ itself is meromorphic at 0, having a simple pole there with residue 1. By the $\mathbb{Z}$-periodicity, the same holds at each integer n. This matches the behavior of $\pi \cot \pi z$. Thus the difference $\pi \cot \pi z - \varphi(z)$ is entire. We want to show that it is the zero function.

The first step is to show that the difference is bounded, making it constant by Liouville's theorem. Since the difference is $\mathbb{Z}$-periodic in the x-direction, it suffices to show that is bounded as $|y| \to \infty$, and for this it suffices to show that each of $\pi \cot \pi z$ and $\varphi(z)$ is individually bounded as $|y| \to \infty$. Compute first that

$$\pi \cot \pi z = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = \pi i \frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} = \pi i + \frac{2\pi i}{e^{2\pi i z} - 1}.$$

Also $|e^{2\pi i z}| = e^{-2\pi y}$, so $\lim_{y \to +\infty} \pi \cot \pi z = -\pi i$ and $\lim_{y \to -\infty} \pi \cot \pi z = \pi i$. On the other hand, suppose now that $z = x + iy$ where $0 \le x < 1$ and $|y| > 1$. Then we have the inequalities $|y| \le |z| \le |y| + 1$ and

$$|z^2 - n^2| = |x^2 - y^2 - n^2 + 2ixy| \ge y^2 + n^2 - x^2 \ge y^2 + n^2 - 1.$$

It follows that

$$|\varphi(z)| \le \frac{1}{|y|} + 2(|y| + 1) \sum_{n=1}^{\infty} \frac{1}{y^2 + n^2 - 1}.$$

Let $\eta = \lfloor |y| \rfloor$. Then

$$\sum_{n=1}^{\infty} \frac{1}{y^2 + n^2 - 1} = \sum_{m=0}^{\infty} \sum_{r=1}^{\eta} \frac{1}{y^2 + (m\eta + r)^2 - 1},$$

and for each $m \ge 0$,

$$\sum_{r=1}^{\eta} \frac{1}{y^2 + (m\eta + r)^2 - 1} \le \frac{\eta}{\eta^2 + (m\eta)^2} = \frac{1}{\eta(1 + m^2)}.$$

This shows that

$$|\varphi(z)| \le \frac{1}{|y|} + 2\frac{|y| + 1}{\lfloor |y| \rfloor} \sum_{m=0}^{\infty} \frac{1}{1 + m^2},$$

and so $\varphi(z)$ is bounded as $|y| \to \infty$ as well.

Thus $\pi \cot \pi z - \varphi(z)$ is constant. To see that the constant is 0, set $z = 1/2$. From before, $\varphi(1/2) = 0$. But also $\pi \cot \pi/2 = 0$, giving the result. And further, termwise differentiability of

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z - n} + \frac{1}{z + n} \right)$$

also gives

$$\pi^2 \csc^2 \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2},$$

now with absolute convergence even without grouping the terms for $\pm n$.

As an application, we compare the power series expansions about $z = 0$ of the two now-known-to-be-equal functions

$$z\varphi(z) \qquad \text{and} \qquad \pi z \cot \pi z.$$

For the first expansion, compute that for $|z| < 1$,

$$\begin{aligned} z\varphi(z) &= 1 + 2z^2 \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \frac{1}{1 - z^2/n^2} \\ &= 1 - 2z^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{\infty} \left(\frac{z}{n}\right)^{2k} = 1 - 2 \sum_{k=0}^{\infty} z^{2k+2} \sum_{n=1}^{\infty} \frac{1}{n^{2k+2}} \\ &= 1 - 2 \sum_{\text{even } k \geq 2} \zeta(k) z^k. \end{aligned}$$

That is, $z\varphi(z)$ is a generating function for the Euler–Riemann zeta function $\zeta(k)$ at positive even values of k. On the other hand, the second expansion is essentially a generating function for the Bernoulli numbers. Again for $|z| < 1$,

$$\begin{aligned} \pi z \cot \pi z &= \pi i z + \frac{2\pi i z}{e^{2\pi i z} - 1} = \pi i z + \sum_{k=0}^{\infty} \frac{B_k}{k!} (2\pi i z)^k \\ &= 1 + \sum_{\text{even } k \geq 2} \frac{(2\pi i)^k B_k}{k!} z^k. \end{aligned}$$

Comparing the two shows expansions gives Euler's famous formula,

$$\zeta(k) = -\frac{1}{2} \cdot \frac{(2\pi i)^k B_k}{k!} \quad \text{for all even } k \geq 2.$$

In particular, this formula combines with the values $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ to give

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945}.$$

Euler's formula for $\zeta(k)$ (even $k \geq 2$) can also be obtained by contour integration techniques, as in our text. The idea is that since

$$\pi \cot \pi z = \frac{1}{z} + \sum_{\text{even } k \geq 2} \frac{(2\pi i)^k B_k}{k!} z^{k-1},$$

it follows that for any even $k \geq 2$,

$$\operatorname{Res}_{z=0} \left(\frac{\pi \cot \pi z}{z^k} \right) = \frac{(2\pi i)^k B_k}{k!}.$$

By contour integration,

$$\frac{(2\pi i)^k B_k}{k!} + 2\zeta(k) = 0,$$

and Euler's formula follows immediately.

Since $\pi \cot \pi z$ is $\mathbb{Z}$-periodic it also has a Fourier series expansion. This is not the same thing as is its Laurent series expansion. Instead, if $z = x + iy$ with $y > 0$ then $|e^{2\pi i z}| = e^{-2\pi y} < 1$, and so

$$\begin{aligned} \pi \cot \pi z &= \pi i + \frac{2\pi i}{e^{2\pi i z} - 1} \\ &= \pi i - 2\pi i \sum_{n=0}^{\infty} q^n, \quad \text{where } q = e^{2\pi i z}. \end{aligned}$$