COMPACTNESS AND UNIFORMITY

1. The Extreme Value Theorem

Because the continuous image of a compact set is compact, a continuous complex-valued function φ on a closed ball B is bounded, meaning that there exists some positive real number c such that

$$|\varphi(z)| \leq c \quad \text{for all } z \in B.$$

Example 1. For application of continuity implying boundedness, work in an environment where

- Ω is a region in $\mathbb{C}$,
- $f : \Omega \longrightarrow \mathbb{C}$ is a differentiable function,
- z is a point of Ω and B is a closed ball about z in Ω.

Define a function

$$\varphi : B \longrightarrow \mathbb{C},$$

where

$$\varphi(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z} & \text{if } \zeta \neq z, \\ f'(z) & \text{if } \zeta = z. \end{cases}$$

This function is continuous because f is differentiable at z. Because B is compact, φ is bounded. Therefore, the integral of φ over a small circle around z in B is bounded by a constant times the length of the circle, with the constant independent of the circle. Taking smaller and smaller circles, the corresponding integrals go to 0. This argument will prove Cauchy's integral representation formula.

2. Compactness and Uniform Continuity

Let K be a compact subset of $\mathbb{R}^n$, and let $f : K \longrightarrow \mathbb{R}^m$ be pointwise continuous. Then f is uniformly continuous.

The topological proof of this fact proceeds as follows. Pointwise continuity says that given $\varepsilon > 0$, for each $x \in K$ there is a $\delta_x > 0$ such that f takes the open δ_x-ball about x into the open $\varepsilon/2$-ball about $f(x)$,

$$f(B(x, \delta_x) \cap K) \subset B(f(x), \varepsilon/2).$$

Cover K with a collection of open balls indexed by the points of K,

$$\{B(x, \delta_x/2) : x \in K\}.$$

Then by compactness, there is a finite subcover,

$$\{B(x_n, \delta_{x_n}/2) : n = 1, \ldots, N\}.$$

Making reference to this finite subcover, let

$$\delta = \min\{\delta_{x_1}/2, \ldots, \delta_{x_N}/2\}.$$

This is the global δ required for uniform continuity.

To see that δ works, consider any two points $x, x' \in K$ such that $|x - x'| < \delta$. The first point x must lie in some $B(x_n, \delta_{x_n}/2)$ since these form a cover of K. That is,

$$|x - x_n| < \delta_{x_n}/2.$$

Consequently,

$$|x_n - x'| \leq |x_n - x| + |x - x'| < \delta_{x_n}/2 + \delta \leq \delta_{x_n}/2 + \delta_{x_n}/2 = \delta_{x_n}.$$

The previous two displays give

$$|x - x_n| < \delta_{x_n} \quad \text{and} \quad |x_n - x'| < \delta_{x_n},$$

and so

$$|f(x) - f(x_n)| < \varepsilon/2 \quad \text{and} \quad |f(x_n) - f(x')| < \varepsilon/2,$$

giving the desired result,

$$|f(x) - f(x')| \leq |f(x) - f(x_n)| + |f(x_n) - f(x')| < \varepsilon.$$

However, this proof is admittedly a bit tricky. An alternative argument may be more intuitive, although it is less general. First, *in Euclidean space, a subset K is compact if and only if every sequence in K has a subsequence that converges in K.* This is essentially the Bolzano–Weierstrass Theorem. Granting this, we can prove that continuity on any compact set is uniform by using sequences rather than an open cover.

The proof proceeds by contradiction. As before, suppose that we have a compact set $K \subset \mathbb{R}^n$ and a continuous function $f : K \longrightarrow \mathbb{R}^m$, but now suppose also that f is not uniformly continuous on K. This means that for some $\varepsilon > 0$ there exists no uniform δ. So in particular no reciprocal positive integer $1/n$ will serve as δ in the definition of uniform continuity. Thus for each $n \in \mathbb{Z}^+$ there exist points x_n and y_n in K such that

$$|x_n - y_n| < 1/n \quad \text{and} \quad |f(x_n) - f(y_n)| \geq \varepsilon. \tag{1}$$

Consider the sequences $\{x_n\}$ and $\{y_n\}$ in K. By the sequential characterization of compactness, $\{x_n\}$ has a convergent subsequence that converges in K. Throw away the rest of the x_n's and throw away the y_n's of corresponding index, reindex the remaining terms of the two sequences, and now $\{x_n\}$ converges to some $p \in K$. Since $|x_n - y_n| < 1/n$ for each n (this remains true after the reindexing), $\{y_n\}$ converges to p as well. So

$$\lim x_n = p = \lim y_n,$$

and now the continuity of f gives

$$\lim f(x_n) = f(p) = \lim f(y_n).$$

This violates the second condition in (1) even though the first condition holds, and so the proof by contradiction is complete.

Example 2. For an application of continuity on compact sets being uniform, work in an environment where

- Ω is a region in $\mathbb{C}$,
- $f : \Omega \longrightarrow \mathbb{C}$ is a continuous function,
- γ is a simple closed curve in Ω such that Ω contains all of its interior,
- z is a point interior to γ and B is a closed ball about z in the interior of γ.

Then the product $B \times \gamma$ is a compact set. Let k be a fixed positive integer, and define a function

$$\varphi^{(k)} : B \times \gamma \longrightarrow \mathbb{C},$$

where

$$\varphi^{(k)}(z', \zeta) = \begin{cases} f(\zeta) \cdot \left(\dfrac{\frac{1}{(\zeta - z')^k} - \frac{1}{(\zeta - z)^k}}{z' - z} \right) & \text{if } z' \neq z, \\ f(\zeta) \cdot \dfrac{k}{(\zeta - z)^{k+1}} & \text{if } z' = z. \end{cases}$$

Here the superscript "(k)" does not denote a derivative, only a reminder that k plays a role in the definition of the function. The point of this casewise-in-z' definition, which can look confusing at first, is that $\varphi^{(k)}$ is continuous. Suggestive evidence of the continuity is that for any fixed ζ, the definition of the z-derivative of $1/(\zeta - z)^k$ says that

$$\lim_{z' \to z} \left(\frac{\frac{1}{(\zeta - z')^k} - \frac{1}{(\zeta - z)^k}}{z' - z} \right) = \frac{\partial}{\partial z} \left(\frac{1}{(\zeta - z)^k} \right) = \frac{k}{(\zeta - z)^{k+1}}.$$

However, this is only suggestive evidence, because ζ is held fixed in the display, whereas we need to show that for any fixed ζ on γ, $\varphi^{(k)}(z', \zeta')$ goes to $\varphi^{(k)}(z, \zeta)$ as (z', ζ') goes to (z, ζ). For $z' = z$ this is clear. For $z' \neq z$ compute

$$\begin{aligned} \varphi^{(k)}(z', \zeta') &= f(\zeta') \cdot \frac{\frac{1}{(\zeta' - z')^k} - \frac{1}{(\zeta' - z)^k}}{z' - z} \\ &= f(\zeta') \cdot \frac{(\zeta' - z)^k - (\zeta' - z')^k}{(z' - z)(\zeta' - z')^k (\zeta' - z)^k} \\ &= f(\zeta') \cdot \frac{\sum_{j=0}^{k-1} (\zeta' - z)^j (\zeta' - z')^{k-1-j}}{(\zeta' - z')^k (\zeta' - z)^k}, \end{aligned}$$

and as (ζ', z') goes to (ζ, z), this goes to $f(\zeta) \cdot k/(\zeta - z)^{k+1} = \varphi^{(k)}(z, \zeta)$. So $\varphi^{(k)}$ is continuous, hence uniformly continuous because its domain is compact. In particular, $\varphi^{(k)}(z', \zeta)$ is within any prescribed closeness to $\varphi^{(k)}(z, \zeta)$ simultaneously for all ζ if z' is close enough to z. We will return to this example later in this handout.

3. Compactness and Uniform Convergence

Let K be a compact subset of $\mathbb{C}$, and let $\{\varphi_n\} : K \longrightarrow \mathbb{C}$ be a sequence of functions that converges pointwise to the function $\varphi : K \longrightarrow \mathbb{C}$. Then the convergence need not be uniform.

The standard counterexample here is the sequence of power functions on the unit interval in the real numbers,

$$\{\varphi_n\} : [0,1] \longrightarrow \mathbb{R}, \quad \varphi_n(x) = x^n, \ n = 1, 2, 3, \ldots$$

whose limit function is discontinuous,

$$\varphi : [0,1] \longrightarrow \mathbb{R}, \quad \varphi(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

To see that the convergence of $\{\varphi_n\}$ to φ is not uniform, reason that for any positive integer n, the polynomial φ_n is continuous, and so for x close enough to 1, $\varphi_n(x)$

is close to $\varphi_n(1) = 1$ and hence far from $\varphi(x) = 0$. More specifically, let $\varepsilon = 1/2$, and for any positive integer n let

$$x_n = (1/2)^{1/n}.$$

Then $0 \leq x_n < 1$, and $\varphi_n(x_n) = 1/2$, $\varphi(x_n) = 0$. That is,

$$|\varphi(x_n) - \varphi_n(x_n)| \geq \varepsilon,$$

and so the convergence is not uniform.

In general, *any uniform limit of continuous functions must again be continuous*, so by this principle as well the convergence in our example cannot be uniform. We will return to this point at the end of this writeup.

Although convergence on compact sets need not be uniform, in our context it often is. For example, consider the partial sums of the geometric series on the complex unit disk,

$$\{\varphi_n\} : D \longrightarrow \mathbb{C}, \quad \varphi_n(z) = \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}, \ n = 0, 1, 2, \ldots$$

and the full geometric series,

$$\varphi : D \longrightarrow \mathbb{C}, \quad \varphi(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1 - z}.$$

Compute that for any $z \in D$,

$$|\varphi(z) - \varphi_n(z)| = \left|\frac{z^{n+1}}{1 - z}\right| \leq \frac{|z|^{n+1}}{1 - |z|}.$$

As z varies freely through some fixed compact subset K of D, its absolute value $|z|$ cannot exceed some maximum $r = r_K$ that is strictly less than 1. Thus

$$|\varphi(z) - \varphi_n(z)| \leq \frac{r^{n+1}}{1 - r} \quad \text{for all } z \in K.$$

The right side is independent of z (though it depends on K), and it goes to 0 as n goes to ∞. This shows that *the geometric series converges uniformly in z on compact subsets of the unit disk*.

Example 3. For a similar example of uniform convergence on compact sets, work in an environment where

- Ω is a region in $\mathbb{C}$,
- $f : \Omega \longrightarrow \mathbb{C}$ is a differentiable (and hence continuous) function,
- γ is a circle in Ω such that Ω contains all of its interior,
- R is the radius of γ, a is the centerpoint of γ, and z is any point interior to γ.

Define a sequence of functions

$$\{\varphi_n\} : \gamma \longrightarrow \mathbb{C}, \quad \varphi_n(\zeta) = f(\zeta) \sum_{k=0}^{n} \frac{(z - a)^k}{(\zeta - a)^{k+1}}, \ n = 0, 1, 2, \ldots$$

Then their pointwise limit function is

$$\varphi : \gamma \longrightarrow \mathbb{C}, \quad \varphi(\zeta) = f(\zeta) \sum_{k=0}^{\infty} \frac{(z - a)^k}{(\zeta - a)^{k+1}}.$$

The claim is that $\{\varphi_n\}$ converges to φ uniformly on γ. To see this, let n be a positive integer and compute that for any $\zeta \in \gamma$,

$$|\varphi(\zeta) - \varphi_n(\zeta)| \le |f(\zeta)| r^n \frac{r}{R(1-r)} \quad \text{where } r = \frac{|z-a|}{R} < 1.$$

The function $|f| : \gamma \longrightarrow \mathbb{R}_{\ge 0}$ is continuous and its domain γ is compact, so its image is compact, meaning that it takes a maximum M. Therefore, for all $\zeta \in \gamma$,

$$|\varphi(\zeta) - \varphi_n(\zeta)| \le M r^n \frac{r}{R(1-r)} = C r^n \text{ where } C > 0 \text{ is a constant.}$$

Since the right side is independent of ζ and it goes to 0 as n goes to ∞, the convergence is uniform in ζ, as claimed. This uniform convergence will justify passing a sum through an integral to obtain the power series representation of an analytic function.

A second way to treat this example is to observe that the functions φ_n converge to φ pointwise, and they are bounded by a constant that is independent of n. At least in the case that γ is piecewise $\mathcal{C}^1$, so that integration over γ reduces to real integration, either the bounded convergence theorem or the Lebesgue dominated convergence theorem lets us assert that the integrals of the functions φ_n over γ tend to the integral of φ over γ, because a constant function is integrable over a curve of finite length.

Example 2, continued. For another example, return to the situation at the end of previous section, where

- Ω is a region in $\mathbb{C}$,
- $f : \Omega \longrightarrow \mathbb{C}$ is a continuous function,
- γ is a simple closed curve in Ω such that Ω contains all of its interior,
- z is a point interior to γ and B is a closed ball about z in the interior of γ,

and for any positive integer k we studied the function

$$\varphi^{(k)} : B \times \gamma \longrightarrow \mathbb{C},$$

given by

$$\varphi^{(k)}(z', \zeta) = \begin{cases} f(\zeta) \cdot \left(\dfrac{\frac{1}{(\zeta - z')^k} - \frac{1}{(\zeta - z)^k}}{z' - z} \right) & \text{if } z' \ne z, \\ f(\zeta) \cdot \dfrac{k}{(\zeta - z)^{k+1}} & \text{if } z' = z. \end{cases}$$

Take a sequence $\{z_n\}$ in the ball B that converges to z, with each z_n distinct from z. Define a corresponding sequence of functions of the one variable ζ,

$$\{\varphi_n^{(k)}\} : \gamma \longrightarrow \mathbb{C}, \quad \varphi_n^{(k)}(\zeta) = \varphi^{(k)}(z_n, \zeta), \ n = 1, 2, 3, \dots,$$

and the corresponding limit function, also of the one variable ζ (reusing the notation $\varphi^{(k)}$ here),

$$\varphi^{(k)} : \gamma \longrightarrow \mathbb{C}, \quad \varphi^{(k)}(\zeta) = \varphi^{(k)}(z, \zeta).$$

In consequence of the discussion at the end of the previous section, the sequence $\{\varphi_n^{(k)}\}$ converges uniformly in ζ to $\varphi^{(k)}$, because

$$\varphi_n^{(k)}(\zeta) - \varphi^{(k)}(\zeta) = \varphi^{(k)}(z_n, \zeta) - \varphi^{(k)}(z, \zeta)$$

and the distance $|(z_n, \zeta) - (z, \zeta)| = |(z_n - z, 0)| = |z_n - z|$ in $B \times \gamma$ is independent of ζ. This makes the quantity in the display small, independently of ζ, by uniform continuity. This uniform convergence will justify passing a limit through an integral to obtain Cauchy's formula for the derivatives of an analytic function.

The uniform convergence can be checked directly by formula as well. The crux is that by some algebra,

$$\begin{aligned}
\frac{\frac{1}{(\zeta-z')^k}-\frac{1}{(\zeta-z)^k}}{z'-z}-\frac{k}{(\zeta-z)^{k+1}} &= \frac{(\zeta-z)^k-(\zeta-z')^k}{(z'-z)(\zeta-z)^k(\zeta'-z)^k}-\frac{k}{(\zeta-z)^{k+1}}\\
&= \frac{\sum_{j=0}^{k-1}(\zeta-z)^j(\zeta-z')^{k-1-j}}{(\zeta-z')^k(\zeta-z)^k}-\frac{k}{(\zeta-z)^{k+1}}\\
&= \frac{\sum_{j=0}^{k-1}(\zeta-z)^{j+1}(\zeta-z')^{-1-j}}{(\zeta-z)^{k+1}}-\frac{k}{(\zeta-z)^{k+1}}\\
&= \frac{1}{(\zeta-z)^{k+1}}\sum_{j=0}^{k-1}\left(\left(\frac{\zeta-z}{\zeta-z'}\right)^{j+1}-1\right)\\
&= \frac{1}{(\zeta-z)^{k+1}}\left(\frac{\zeta-z}{\zeta-z'}-1\right)\sum_{j=0}^{k-1}\sum_{i=0}^{j}\left(\frac{\zeta-z}{\zeta-z'}\right)^i\\
&= (z'-z)\frac{1}{(\zeta-z)^{k+1}(\zeta-z')}\sum_{j=0}^{k-1}\sum_{i=0}^{j}\left(\frac{\zeta-z}{\zeta-z'}\right)^i.
\end{aligned}$$

Thus we have

$$\begin{aligned}
|\varphi_n^{(k)}(\zeta)-\varphi^{(k)}(\zeta)| &= \left|f(\zeta)\cdot(z_n-z)\frac{1}{(\zeta-z)^{k+1}(\zeta-z_n)}\sum_{j=0}^{k-1}\sum_{i=0}^{j}\left(\frac{\zeta-z}{\zeta-z_n}\right)^i\right|\\
&\le c|z_n-z|,
\end{aligned}$$

for a positive constant c that is independent of ζ as soon as z_n is close enough to z. That is, as n grows, $\varphi_n^{(k)}(\zeta)$ goes to $\varphi^{(k)}(\zeta)$ at a rate independent of ζ. This is the uniform convergence. The clutter of algebra here perhaps makes us appreciate how uniform continuity, in consequence of compactness, lets us establish it without having to work through this algebra.

A third way to treat the same example is to observe that because the two-variable function $\varphi^{(k)}$ is continuous on its compact domain, the one-variable functions $\varphi_n^{(k)}$ converge to the one-variable function $\varphi^{(k)}$ pointwise, and they are bounded by a constant that is independent of n. At least in the case that γ is piecewise $\mathcal{C}^1$, so that integration over γ reduces to real integration, either the bounded convergence theorem or the Lebesgue dominated convergence theorem lets us assert that the integrals of the one-variable functions $\varphi_n^{(k)}$ over γ tend to the integral of the one-variable function $\varphi^{(k)}$ over γ.

As mentioned above, any uniform limit of continuous functions must again be continuous. That is,

Let S be a subset of $\mathbb{C}$. Consider a sequence of continuous functions on S,

$$\{\varphi_0,\varphi_1,\varphi_2,\dots\}:S\longrightarrow\mathbb{C}.$$

Suppose that the sequence converges uniformly on S to a limit function

$$\varphi:S\longrightarrow\mathbb{C}.$$

Then φ is also continuous.

This is shown by a three-epsilon argument, as follows. For any $z, \tilde{z} \in S$ and any $n \in \mathbb{N}$,

$$|\varphi(\tilde{z}) - \varphi(z)| \le |\varphi(\tilde{z}) - \varphi_n(\tilde{z})| + |\varphi_n(\tilde{z}) - \varphi_n(z)| + |\varphi_n(z) - \varphi(z)|.$$

Let $\varepsilon > 0$ be given. Since $\{\varphi_n\}$ converges to φ uniformly on S it follows that for all large enough indices n, independently of z and $\tilde{z}$,

$$|\varphi(\tilde{z}) - \varphi_n(\tilde{z})| < \varepsilon \quad \text{and} \quad |\varphi_n(z) - \varphi(z)| < \varepsilon.$$

Fix any such index n, and fix the point $z \in S$. Since φ_n is continuous at z there exists some $\delta = \delta(\varepsilon, z) > 0$ such that

$$|\tilde{z} - z| < \delta \implies |\varphi_n(\tilde{z}) - \varphi_n(z)| < \varepsilon.$$

It follows from these considerations that

$$|\tilde{z} - z| < \delta \implies |\varphi(\tilde{z}) - \varphi(z)| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

This completes the proof that φ is continuous.